# Some Problems In The Theory Of Riemann Summability Of Infinite Series

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BY

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## CERTIFICATE

This is to certify that the contents of this thesis entitled, SOME PROBLEMS IN THE THEORY OF RIEMANN SUMMABILITY OF INFINITE SERIES, is an original research work of Mr. Vinod K. Parasher, done under my supervision.

I further certify that the work of this thesis, either partly or fully has not been submitted to any other institution for the award of any other degree.

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## SUMMARY

1. __Definitions and Notations__ : Let $\Sigma a_n$ be a given infinite series with $s_n$ as its n-th partial sum. Let $\{p_n\}$ be a sequence of constants, real or complex, and

$$P_n = \sum_{v=0}^{n} p_v \neq 0, \quad P_{-1} = p_{-1} = 0.$$

The sequence-to-sequence transformation :

$$t_n = (P_n)^{-1} \sum_{v=0}^{n} p_{n-v} s_v$$

defines the sequence $\{t_n\}$ of Nörlund means of $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$ .

The series $\Sigma a_n$ , or the sequence $\{s_n\}$ , is defined to be summable by the Nörlund method, or summable $(N, p_n)$ to sum s (finite), if $\lim_{n\to\infty} t_n = s$ .

Let us write

$$(*) \qquad f(x) = \Sigma a_n x^n .$$

Then the series $\Sigma a_n$ is said to be absolutely Abel summable, or summable $|A|$, if the series on the right of $(*)$ converges, for $0 \leq x < 1$, and its sum $f(x) \in BV\ (0, 1)$.

We say that the series $\Sigma$ am is absolutely Abel summable with index k $(k \geq 1)$, or simply summable $|A|_k$, if the series on the right of $(*)$ converges for $0 \leq x < 1$ and

$$\int_0^1 (1-x)^{k-1} \, |f'(x)|^k \, dx < \infty .$$

Let $f_p(x) = (\sin x/x)^p \quad (x \neq 0), \quad f_p(0) = 1$

The series $\Sigma a_n$ is said to be summable $(R, p)$ to sum s, p being a positive integer, if the series

$$F_p(t) = \sum_{n=1}^{\infty} a_n f_p(nt)$$

converges in some interval $0 < t < t_0$, and $F_p(t) \to s$, as $t \to 0$.

The series $\Sigma a_n$ is said to be summable absolutely by Riemann method of order p, or simply summable $|R, p|$, where p is a positive integer, if the series

$$(**) \qquad F_p(x) = \sum_{k=1}^{\infty} a_k f_p(kx)$$

is convergent for $x \in (0, \delta)$, $\delta > 0$ and $F_p(x) \in BV\ (0, \delta)$, that *is,*

$$\int_0^\delta \left|\frac{d}{dx}(F_p(x))\right| dx < \infty .$$

We say that the series $\Sigma a_n$ is absolutely summable by Riemann method of order p and index k, or simply summable $|R, p|_k$, $k \geq 1$, when p is a positive integer, if the series (**) is convergent for $x \in (0, \delta)$, $\delta > 0$, and

$$\int_0^\delta x^{k-1} \left|\frac{d}{dx}(F_p(x))\right|^k dx < \infty .$$

The method $|R, p|_1$ is the same as the method $|R, p|$. For $k > 1$ the methods $|R, p|$ and $|R, p|_k$ are independent.

Let $S_n^\alpha$ be the n-th Cesàro-sum of order $\alpha$ ($\alpha > -1$) of the sequence $\{s_n\}$, defined by;

$$S_n^\alpha = \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v .$$

Then the series $\Sigma a_n$ is said to be summable by the Riemann-Cesàro method of order p and index $\alpha$ ($-1 \leq \alpha \leq p-1$), or briefly summable $(R, p, \alpha)$ to sum s, if the series

$$F_p(\alpha, t) = (C_{p,\alpha})^{-1}\, t^{\alpha+1} \sum_{n=1}^{\infty} S_n^\alpha f_p(nt)$$

where

$$C_{p,\alpha} = \begin{cases} \dfrac{1}{\Gamma(\alpha+1)} \displaystyle\int_0^{\infty} u^{\alpha-p}(\sin u)^p\, du & , -1<\alpha<p-1; \\ \pi/2 & , \alpha=0,\ p=1; \\ 1 & , \alpha=-1; \end{cases}$$

converges in some interval $0 < t < t_o$ and $F_p(\alpha,t) \to s$, as $t \to 0$.

When $\alpha = -1$, $0$ and $p = 1$, then the $(R, p, \alpha)$-method reduces to $(R, 1)$ and $(R_1)$-summability methods respectively.

The series $\Sigma\, a_n$ is said to be summable $(K, 1, \alpha)$ to the sum $s$, if the series

$$K(\alpha,t) = B_\alpha^{-1}\, t^{\alpha+1} \sum_{n=1}^{\infty} s_n^{\alpha} \int_t^{\pi} \frac{\sin nx}{2\tan x/2}\, dx ,$$

converges in some interval $0 < t < t_o$, and if $\lim_{t\to o} K(\alpha,t) = s$, where

$$B_\alpha = \begin{cases} \pi/2 & , \alpha = -1; \\ (\alpha+1)^{-1} \sin(\alpha+1)\pi/2 & , -1 < \alpha < 0; \\ 1 & , \alpha = 0 . \end{cases}$$

when $\alpha = -1$, the method $(K, 1, \alpha)$ reduces to the method $(K,1)$.

Let $p > 0$, $0 \leq \lambda_0 < \lambda_1 < \ldots < \lambda_n \to \infty$.

If the series

$$R_\lambda^p(t) = \sum_{n=0}^{\infty} a_n f_p(\lambda_n t),$$

converges in some interval $0 < t < t_0$ and $R_\lambda^p(t) \to s$, as $t \to 0$, then the series $\Sigma a_n$ is said to be summable $(R, \lambda_n, p)$ to sum $s$.

A method of summation is said to be regular, if, it assign to every convergent series its actual value. If it furthermore assigns to every series the values $+\infty$ which diverges to $+\infty$, it is said to be totally regular.

2. The present Thesis consists of six chapters. Chapter I is introductory and it also contains a résumé of hitherto known results which have direct interconnection with our investigations.

In Chapter II, we prove the following theorems :

Theorem 1. If $\Sigma a_n$ is $(N, p_n)$-summable and if

$$\sigma_n = \sum_{v=1}^{n} |T_v - T_{v-1}| = O(P_n),$$

then $\Sigma a_n$ is $(\bar{R}, 1)$-summable, provided $\{p_n\}$ is non-negative, non-increasing sequence such that $P_n \to \infty$, and

(i) $d_n = \sum_{v=0}^{n} c_v = O(\frac{1}{P_n})$ ;

(ii) $\sum_{v=n+1}^{\infty} |c_v| = O(\frac{1}{P_n})$ , for $n \geq 0$ ;

(iii) $\sum_{v=n}^{\infty} \frac{P_{v-n}}{v(v+1)} = O(\frac{P_n}{n})$ ;

(iv) $\sum_{v=0}^{n} \frac{1}{P_v} = O(\frac{n}{P_n})$ ;

(v) for a positive integer $\mu$ and $n = [\mu t^{-1}]$, $\tau = [t^{-1}]$,

$$P_n = O(P_\mu P_\tau).$$

Theorem 2. Under the hypotheses of Theorem 1, $\Sigma a_n$ is summable $(R_1)$.

We observe that these theorems generalise certain results of Szász [(5), (6)], Varshney (7), and Singh (4).

We also deduce from our theorems two corollaries, which are easy to apply.

<u>Corollary I.</u> <u>If</u> $\Sigma a_n$ is $(N, p_n)$-<u>summable and if</u>

$$\sigma_n = \sum_{\nu=1}^{n} |T_\nu - T_{\nu-1}| = O(P_n),$$

<u>then</u> $\Sigma a_n$ <u>is</u> $(N, 1)$-<u>summable, provided</u> $\{p_n\}$ <u>is non-negative non-increasing sequence, such that,</u> $p_0 = 1$, $P_n \to \infty$, <u>and</u>

(i) $\left\{\frac{p_{n+1}}{p_n}\right\}$ <u>is non-decreasing</u> ;

(ii) $\sum_{\nu=n}^{\infty} \frac{p_{\nu-n}}{\nu(\nu+1)} = O(\frac{p_n}{n})$ ;

(iii) $\sum_{\nu=0}^{n} \frac{1}{P_\nu} = O(\frac{n}{P_n})$ ;

(iv) <u>for a positive integer</u> $\mu$ <u>and</u> $n = [\mu t^{-1}]$, $\tau = [t^{-1}]$,

$$P_n = O(P_\mu P_\tau).$$

<u>Corollary II. Under the hypotheses of Corollary</u> I, $\Sigma a_n$ <u>is summable</u> $(R_1)$.

In Chapter III we have considered the same problem as above for summability $(R, 1, \alpha)$, and the following theorems have been proved, which include the above results as special cases.

<u>Theorem 3. If</u> $\Sigma a_n$ <u>is</u> $(N, p_n)$<u>-summable, and if</u>

$$\text{(i)} \qquad \sigma_n = \sum_{v=1}^{n} |T_v - T_{v-1}| = O(P_n) ,$$

<u>then the series</u> $\Sigma a_n$ <u>is summable by</u> $(R, 1, \alpha)$<u>-method for</u> $-1 \leq \alpha \leq 0$, <u>provided that</u> $\{p_n\}$ <u>is a non-negative, non-increasing sequence such that</u> $P_n \to \infty$, <u>and</u>

$$\text{(ii)} \qquad d_n = \sum_{v=0}^{n} c_v = O\left(\frac{1}{P_n}\right) ;$$

$$\text{(iii)} \qquad \sum_{v=n+1}^{\infty} |c_v| = O\left(\frac{1}{P_n}\right) , \quad n \geq 0 ;$$

$$\text{(iv)} \qquad \sum_{v=n}^{\infty} \frac{P_{v-n}}{v(v+1)} = O\left(\frac{P_n}{n}\right) ,$$

$$\text{(v)} \qquad \sum_{v=0}^{n} \frac{1}{P_v} = O\left(\frac{n}{P_n}\right) ;$$

(vi) <u>for a positive integer</u> $\mu$ <u>and</u> $n = [\mu t^{-1}]$, $\tau = [t^{-1}]$,

$$P_n = O(P_\mu P_\tau).$$

Theorem 4. *Let* $\{p_n\}$ *be a positive, non-increasing sequence such that* $p_0 = 1$, $P_n \to \infty$, $\{p_{n+1}/p_n\}$ *is a non-decreasing sequence and the conditions* (iv) *through* (vi) *hold. If* $\Sigma a_n$ *is* $(N, p_n)$*-summable, and if* (i) *holds, then* $\Sigma a_n$ *is also summable* $(R, 1, \alpha)$, *for* $-1 \leq \alpha \leq 0$.

Since $(R, 1, \alpha)$-method of summation has some properties similar to that of $(K, 1, \alpha)$-method of summation, therefore, in Chapter IV we have established the following theorems for $(K, 1, \alpha)$-summability methods which are analogous to preceding ones.

Theorem 5. *Under the hypotheses of Theorem* 3, $\Sigma a_n$ *is summable* $(K, 1, \alpha)$ *for* $-1 \leq \alpha \leq 0$.

Theorem 6. *Under the hypotheses of Theorem* 4, $\Sigma a_n$ *is summable* $(K, 1, \alpha)$ *for* $-1 \leq \alpha \leq 0$.

Chapter V deals with the problem on total regularity which have also been previously studied by Lee (3) and Hirokawa (2). In this chapter, we have established the following results :

Theorem 7. *The method* $(R, 2p+1, \alpha)$, $0 \leq \alpha < 2p$, $p \geq 1$ *is not totally regular.*

<u>Theorem 8. Let</u> $p = 1, 2, \ldots$ <u>and let</u> $\Lambda_n = \lambda_{n+1}/(\lambda_{n+1}-\lambda_n)$, $0 \le \lambda_0 < \lambda_1 < \ldots < \lambda_n \to \infty$ <u>such that</u> $\sum_{n=v}^{\infty} \frac{1}{\Lambda_n \lambda_n^p} = O(\frac{1}{\lambda_v^p})$, <u>and for every sequence</u> $\{t_v\}$ <u>of positive numbers tending to zero and integers</u> $N_v$, $\lim_{v\to\infty} \lambda N_v t_v = \pi$. <u>If</u> $a_n \ge -K/\Lambda_n$ $(n=1,2,\ldots, K$ ; <u>positive constant</u>), $\sum_{n=1}^{\infty} a_n f_p(\lambda_n t)$ <u>converges in</u> $0 \le t \le t_0$. <u>Then</u> $\sum_{n=1}^{\infty} a_n = +\infty$ <u>implies</u>

$$\lim_{t\to 0} \sum_{n=1}^{\infty} a_n f_p(\lambda_n t) = +\infty.$$

In the sixth and the last chapter we define absolute Riemann summability of order p, for an index k, $k \ge 1$, and proved the following theorem which establishes a relation between $|A|_k$ and $|R, p|_k$ methods of summability.

<u>Theorem 9. Let</u> $k \ge 1$. <u>If the series</u> $\Sigma a_n$ <u>is summable</u> $|R, p|_k$ <u>and if</u> $a_n = O(1)$, <u>then it is also summable</u> $|A|_k$

It is to remark that this theorem generalises a recent result of Gosberg (1) , concerning absolute Abel and absolute Riemann summability.

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## PREFACE

The present Thesis entitled, SOME PROBLEMS IN THE THEORY OF RIEMANN SUMMABILITY OF INFINITE SERIES, is the outcome of my researches that I have been pursuing since 1969, under the esteemed supervision of Dr. Z.U.Ahmad, M.Sc., D.Phil., D.Sc., Reader, Department of Mathematics and Statistics, Aligarh Muslim University, Aligarh.

It has been my proud privilege to have accomplished my researches under the able supervision of Dr. Z.U.Ahmad, who has made substantial contributions in the field of Absolute summability. I have great pleasure in taking this opportunity of acknowledging my deep sense of gratitude and indebtedness to Dr. Ahmad for his inspiring guidance and encouragement throughout the course of these researches.

The Thesis consists of six chapters. In first chapter, we give a résumé of hitherto known results which have interconnections with our investigations. Chapter II concerns with the study of a relation of Nörlund summability with Lebesgue and $(R_1)$ summability methods, while Chapter III and IV center around the study of the same problem for

Riemann-Cesàro summability and $(K, 1, \alpha)$-summability respectively. Chapter V deals with the study of certain problems on total regularity of Riemann-Cesàro summability and $(R, \lambda_n, p)$-summability methods. The last chapter (Chapter VI), contains a result in which we have discussed a problem concerning a relation between absolute Abel and absolute Riemann summability for an index k, $k \geq 1$. Towards the end a comprehensive bibliography of various publications referred to in the body of the Thesis, has been given.

Some portion of the Thesis has already been accepted in some of the Indian and European journals. I attach herewith the attested copies of letters of acceptance in the Appendix. Further, I have to add that one of the results of Chapter II has been presented by me at the 59th Session of the Indian Science Congress Association, 1972, and has been abstracted in its Proceedings, Part III, and the other result has been announced in the August issue of the Notices of the American Mathematical Society.

I owe a great deal to Professor M.A.Kazim and Professor S.I.Husain, of the Department of Mathematics and Statistics, Aligarh Muslim University, for their constant encouragement during my researches.

Aligarh

December 27, 1973.

Vinod Kumar Parasher

CONTENTS

## CONVENTIONS USED IN THE THESIS

### SUMMATION CONVENTION :

$\Sigma$ written without limits, usually denotes $\sum\limits_{0}^{\infty}$ or $\sum\limits_{1}^{\infty}$ if a term of zero is not defined.

$\sum\limits_{n} a_n$ is the sum of all $a_n$'s which are defined.

### BINOMIAL COEFFICIENTS :

For $n=0, 1, 2, \dots$ , $A_n^{\alpha}$ is defined by the identity :

$$\sum A_n^{\alpha} x^n = (1-x)^{-\alpha-1} \qquad ( |x| < 1 ).$$

$$A_n^{\alpha} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} \qquad ( \alpha > -1 ).$$

$$A_n^{-\alpha} = 0 \qquad ( n > \alpha+1,\ \alpha = 1, 2, \dots )$$

### FINITE DIFFERENCES :

For any function $f(n, v)$ ,

$$\Delta_v f(n,v) = f(n,v) - f(n,v+1); \quad \Delta_v^{0} f(n,v) = f(n,v) ,$$

$$\Delta_v^m f(n,v) = \Delta\left( \Delta^{m-1} f(n,v) \right) \qquad ( m = 1, 2, \ldots ).$$

For any sequence $\{s_n\}$,

$$\Delta s_n = s_n - s_{n+1} , \quad \Delta^0 s_n = s_n ,$$

$$\Delta^k s_n = \Delta \left( \Delta^{k-1} s_n \right) \quad (k = 1, 2, \ldots )$$

$$= \sum_{j=0}^{k} \binom{k}{j} (-1)^j s_{n+j} .$$

CONSTANT :

K denotes an absolute constant independent of the variable under consideration, but is not necessarily the same at each occurrence.

$O$, $o$ and $\sim$.

If $g > 0$, then

$f = O(g)$ means $|f| < K\ |g|$ ;

$f = o(g)$ mean $f/g \to 0$ .

The notation $\sim$ is used in two different context :

(i) If P and Q are two equivalent summability processes, then we write

$$P \sim Q .$$

Similarly we interpret

$$|P| \sim |Q| .$$

(ii) If $f$ and $g$ represent two functions, then

$$f \sim g \quad \text{means} \quad f/g \to 1 .$$

BOUNDED VARIATION :

By ' $\{f_n\} \in BV$', we mean that the sequence $\{f_n\}$ is of bounded variation, that is to say ,

$$\Sigma \; |f_n - f_{n-1}| \leq K .$$

By ' $f(x) \in BV\,(h, k)$ ', we mean that $f(x)$ is a function of bounded variation in the interval $(h, k)$.

INTEGRAL PART OF x :

$[x]$ denotes the algebrically greatest integer not exceeding $x$ .

Apart from these, all notations and conventions of Chapter I will be adhered to throughout the rest of the Thesis without specific mention, unless otherwise stated.

# Chapter I

## INTRODUCTION

1.1 With the publication of Cauchy's[1] historic treatise, 'Cours d'Analyse Algébrique' and by the genious researches of Abel,[2] were laid the foundations of a vigorous theory of infinite series. Although the principle of convergence (sometimes for precision called Cauchy convergence) clearly divided infinite series into two classes, viz., those which have a finite (and unique) sum in the sense of Cauchy and those that fail to have, there remained to be precisely apprehended the distinction between properly divergent series and series with finitely oscillatory partial sums. Abel wrote in 1828 : 'Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever' . But, towards the close of the last century a large variety of oscillatory series were

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1) Cauchy (1).

2) Abel (1).

brought within the framework of a sound mathematical interpretation through the concept of summability.

Summability is a generalization of the notion of Cauchy convergence [1] in the sense that the partial sum is to be replaced by a suitable average of it in a certain prescribed manner. For the pioneering studies that led to the formulation of the theory of summability, credit goes 'inter alia' to Hölder, Cesàro, Riemann, Hausdorff, Borel and others. [2]

Analogously, there emerged the concept of Absolute Summability [3] as a natural generalization of the notion of absolute convergence. A series although not absolutely convergent in the classical sense yet might be absolutely summable in an appropriate sense.

1.2 Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series, with $\{s_n\}$ as the sequence of its partial sums. Broadly speaking, commonly used methods of summability fall into one or the

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1) Hobson (1). p.65.

2) cf. Hardy (1); see also Ishiguro (1), (2).

3) The earliest work known to us is that of Fekete (1).

other of the two categories, viz., T-methods based upon the formation of a sequence of auxiliary means defined by sequence-to-sequence transformation :

$$(1.2.1) \qquad t_n = \sum_k c_{n,k}\, s_k \qquad ( n= 0,1,2\ldots ),$$

$c_{n,k}$ being the element of the n-th row and k-th column of the matrix $\| T \| = (c_{n,k})$, the matrix of summability ; $\emptyset$-methods based upon the formation of a functional transformation defined either by the sequence-to-function transformation :

$$(1.2.2) \qquad t(x) = \sum_k \emptyset_k(x)\ s_k\ ,$$

or by the series-to-function transformation :

$$(1.2.3) \qquad t(x) = \sum_k \bar{\emptyset}_k(x)\ a_k\ ,$$

where x is a continuous parameter, and $\emptyset_k(x)$ and $\bar{\emptyset}_k(x)$ are defined over an appropriate interval of x.

There are also other types of transformation belonging to the T and $\emptyset$-categories, with which we are not concerned here.

The series $\Sigma\, a_n$ , or the sequence $\{s_n\}$ is said to be summable to a finite number s by a T-method or a Ø-method according as the sequence $\{t_n\}$, or the function $t(x)$, tends to s, as n tends to infinity or x tends to the appropriate limit, depending upon the method.[1)]

The series $\Sigma a_n$ is said to be absolutely convergent if $\Sigma |a_n| < \infty$, that is, if

(1.2.4) $$\sum_n | s_n - s_{n-1} | < \infty .$$

The interpretation of the phenomenon (1.2.4) as the bounded variation of the sequence $\{s_n\}$ laid the foundation of the structure of absolute summability. More precisely, the series $\Sigma a_n$, or the sequence $\{s_n\}$, is said to be absolutely summable to the sum s, by a T-method or a Ø-method, according as the sequence $\{t_n\}$ or the function $t(x)$ is of bounded variation as a sequence or as a function over the relevant interval of x respectively, and further $t_n \to s$ as $n \to \infty$ or $t(x) \to s$ as x tends to a suitable limit.

It should be noted that absolute convergence implies convergence.

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1) Knopp (1). p.474.

1.3 The sequence-to-sequence transformation (1.2.1) is said to be conservative (or absolutely conservative) if the convergence (or absolute convergence) of the sequence $\{s_n\}$ implies that of the sequence $\{t_n\}$ in each case, and is said to be regular (or absolutely regular), if further

$$\lim_{n\to\infty} s_n = s \implies \lim_{n\to\infty} t_n = s .$$

Morley [1] has shown that an absolutely conservative transformation is not necessarily conservative.

The necessary and sufficient conditions that the transformation (1.2.1) should be conservative, are : [2]

$$(1.3.1)\qquad \begin{cases} \text{(i)} \quad \lim_{n\to\infty} c_{n,k} = \delta_k & (k = 0,1,\ldots), \\ \text{(ii)} \quad \lim_{n\to\infty} \sum_{k=0}^{\infty} c_{n,k} = \delta , & \\ \text{(iii)} \quad \sum_{k}^{\infty} |c_{n,k}| \leq K & (n=0,1,\ldots), \end{cases}$$

where $K$ is a constant independent of $n$.

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1) Morley (1).

2) Hardy (1), Theorem 1.

If, in addition, $\delta_k = 0$ for each k and $\delta=1$, then (1.3.1) gives the necessary and sufficient conditions for the transformation to be regular. [1]

The necessary and sufficient conditions that the transformation (1.2.1) should be absolutely conservative, are :

$$(1.3.2)\quad \begin{cases} \text{(i)} \quad \sum_{k=0}^{\infty} c_{n,k} \quad \text{converges for each } n, \\ \\ \text{(ii)} \quad \sum_{n=0}^{\infty} \sum_{k=p}^{\infty} (c_{n,k} - c_{n-1,k}) \leq K \;(p=0,1\ldots), \end{cases}$$

where K is a constant independent of $p$, (1.3.2) implies the existence of the limits :

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} c_{n,k} = a\ ,$$

$$\lim_{n\to\infty} c_{n,k} = a_k \qquad\qquad (k= 0,1,\ldots\ ).$$

The transformation (1.2.1) is absolutely regular [2] if,

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1) Hardy (1), Theorem 2.

2) This was first proved by Mears(1). A short proof was subsequently supplied by Knopp and Lorentz(1). Also see Sunouchi(1).

$$\text{(iii)} \quad \alpha = 1, \quad \alpha_k = 0 \qquad (k = 0, 1, \ldots).$$

A sequence-to-sequence method of summability C, or the transformation (1.2.1) associated with it, is said to be totally regular, if $\{t_n\}$ satisfies the following :

$$\lim_{n\to\infty} s_n = s \qquad (-\infty \le s \le +\infty)$$

implies

$$\lim_{n\to\infty} t_n = s \qquad (-\infty \le s \le +\infty).$$

In particular, it is known that a real positive transformation is totally regular whenever it is regular. 1)

1.4 The transformations (1.2.2) or (1.2.3) is said to be conservative (absolutely conservative) if convergence (bounded variation) of the sequence $\{s_n\}$ implies that $t(x)$ tends to a finite limit as $x$ tends to a suitable limit ($t(x) \in BV$ over suitable range of $x$), and it is said to be regular (absolutely regular), if in addition

$$\lim_{n\to\infty} s_n = s \quad \Rightarrow \quad \lim_{x\to a} t(x) = s.$$

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1) cf. Hardy(1), p.52; see also W.A.Hurwitz(1).

The necessary and sufficient conditions that the transformation (1.2.2) should be conservative, are:[1]

$$(1.4.1)\quad \begin{cases} \text{(i)} & \sum_{k=1}^{\infty} \phi_k(x) \leq M, \text{ independently of } x > x_0 \text{ ,} \\ \text{(ii)} & \lim_{x\to\infty} \phi_k(x) = \alpha_k, \text{ for every fixed } k, \\ \text{(iii)} & \sum_{k=1}^{\infty} \phi_k(x) = \Phi_k(x) \to \alpha \text{ , as } x \to \infty . \end{cases}$$

If, in addition, $\alpha_k = 0$ for every fixed k, and $\alpha = 1$, the conditions (1.4.1) are necessary and sufficient for the transformation (1.2.2) to be regular. [2]

The necessary and sufficient conditions that the transformation (1.2.3) should be conservative, are : [3]

$$(1.4.2)\quad \begin{cases} \text{(i)} & \sum_{k=1}^{\infty} \bar{\phi}_k(x) - \bar{\phi}_{k+1}(x) \leq M \text{ for every } x > x_0 \\ \text{(ii)} & \lim_{x\to\infty} \bar{\phi}_k(x) = \beta_k \text{ , for every fixed } k. \end{cases}$$

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1) Cooke(1), p.60, Theorem (4.1,I), where other references are given. See also Dienes (1).

2) Cooke (1), p.64, Theorem (4.1, II ).

3) Cooke (1), p.66, Theorem (4.2,I).

If, in addition, $\beta_k = 1$, for every fixed k, then the conditions (1.4.2) are necessary and sufficient for the transformation (1.2.3) to be regular. [1)]

Similarly, we have necessary and sufficient conditions for absolute conservativeness (absolute regularity) of the transformations (1.2.2) and (1.2.3). [2)]

The total regularity of the transformations (1.2.2) and (1.2.3) are defined in the same manner as that for the transformation (1.2.1) of Section 1.3 and the necessary and sufficient conditions for the total regularity of the transformations (1.2.2) and (1.2.3) are also known. [3)]

1.5 Given the summability (or absolute summability) processes P and Q, P is said to be included in Q, or Q to be inclusive of P, if every sequence summable (or absolutely summable) by P is summable (or absolutely summable) by Q, symbolically, $P \subset Q$.

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1) Cooke (1), p.68, Theorem (4.2,II).

2) See Suneuchi (1).

3) See H.Hurwitz (1) and W.A. Hurwitz (1).

If $P \subset Q$ and $Q \subset P$ , then the two processes are equivalent and this we represent symbolically by $P \sim Q$ .

In the case in which $P \subseteq Q$ and $Q \subseteq P$ are false, we say that the methods P and Q are incomparable.

In the case in which $P \subseteq Q$ ; but $Q \subseteq P$ is false, that is, $P \subset Q$, then the question arises : Would it be possible in some manner to restrict the order of magnitude of terms of the series $\Sigma a_n$ so that, for it $Q \subset P$ (and in effect $P \sim Q$)? The result answering this question in the affirmative is called 'Tauberian' . A result of the type : $P \subseteq Q$ or $P \subset Q$ is called 'Abelian' .

Our aim in the present Thesis is to study the questions of this type.

1.6 Before going to the details of the background against which the problems considered in the present Thesis suggest themselves, and giving a brief résumé of allied results hitherto availabe, we consider it desirable to present here definitions and notations concerning the summability methods that are involved in the present work.

Cesàro summability : Let $S_n^{\alpha}$ be the n-th Cesàro-sum of order $\alpha$ ( $\alpha > -1$)[1] of a given series $\Sigma a_n$, with the sequence of partial sums $\{s_n\}$, defined by the identity :

$$(1.6.1) \qquad S_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha} a_\nu ,$$

where $A_n^\alpha$ being given by :

$$(1.6.2) \qquad \sum_{n=0}^{\infty} A_n^\alpha x^n = (1-x)^{-\alpha-1} , \qquad ( |x| < 1 ).$$

Then the n-th Cesàro mean $s_n^\alpha$ of order $\alpha$ of $\{s_n\}$ is given by:

$$s_n^\alpha = S_n^\alpha / A_n^\alpha .$$

The series $\Sigma a_n$, or the sequence $\{s_n\}$, is said to be summable $(C, \alpha)$, $\alpha > -1$, to sum $s$, if $\lim_{n\to\infty} s_n^\alpha = s$.

It is clear from the definition that summability (C, 0) is the same as convergence.

It is known that $(C, \alpha) \subset (C, \alpha')$, for $\alpha' > \alpha$. In particular, the method $(C, \alpha)$ is regular for $\alpha > 0$.[2]

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1) The justification of the restriction $\alpha > -1$ is indicated, e.g., in Hardy (1), p.97-98.

2) Hardy (1), p.101, Theorem 44 .

<u>Nörlund summability</u> . Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write :

$$P_n = p_0 + p_1 + \ldots + p_n , \quad P_{-1} = p_{-1} = 0.$$

The sequence-to-sequence transformation :

$$(1.6.3) \qquad t_n = ( P_n)^{-1} \sum_{v=0}^{n} p_{n-v} s_v , \qquad (P_n \neq 0),$$

defines the sequence $\{t_n\}$ of Nörlund means [1)] of $\{s_n\}$ , generated by the sequence of coefficients $p_n$ .

The series $\Sigma a_n$ , or the sequence $\{s_n\}$ , is said to be summable by the Nörlund method, or summable $(N, p_n)$ to sum s (finite), if $\lim_{n\to\infty} t_n = s.$ [2)]

The following observations may be made about the nature of the sequence $\{p_n\}$ .

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1) Nörlund(1). A definition substantially the same as that of Nörlund was given by G.F.Woronoi (1) in the Proceedings of the eleventh congress of Russian naturalists and scientists, St. Petersburg, 1902, 60-61 (Russian). It remained unnoticed till it was translated by J.D.Tamarkin in 1932. Reference might be made to Tamarkin (1).

2) cf. Hardy(1), p.64.

(a) If $p_0 = 1$ and $p_n = 0$ $(n \geq 1)$, then summability $(N, p_n)$ is the same as convergence.

(b) If $p_n = A_n^{\alpha-1}$, $\alpha > -1$, then the corresponding Nörlund mean reduces to the familiar $(C, \alpha)$-mean.

(c) In case $p_n = 1/(n+1)$ $(n \geq 0)$, and hence $P_n \sim \log(n+1)$, as $n \to \infty$, the summability $(N, p_n)$ is called harmonic summability.

It is worth pointing out that the most significant trait of summability $(N, p_n)$, apart from the generalization it provides of the Cesaro methods, lies in the fact that itcovers harmonic summability which forms an intermediary between convergence and Cesaro summability of positive order in the scale of summability methods in the view of the inclusions.[1]

$$( C, 0 ) \subset ( N, 1/(n+1) ) \subset (C, r), \quad r > 0 .$$

Necessary and sufficient conditions for the regularity of the Nörlund mean, are:[2]

$$p_n = o( |P_n| ) \quad , \quad \text{as } n \to \infty, \tag{1.6.4}$$

$$\sum_{\nu=0}^{n} |p_\nu| = O(|P_n|) \; , \quad \text{as } n \to \infty. \tag{1.6.5}$$

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1) Hardy (1) ; McFadden (1).

2) Hardy (1), p.65.

Abel summability : The series $\Sigma a_n$ is said to be Abel summable to sum $s$ or summable (A) to $s$, if the power series $\Sigma a_n x^n$ is convergent in $0 \leq x < 1$, and its sum function $f(x)$ tends to a finite limit $s$, as $x \to 1-0$.

In 1826, Abel[1] proved his classical theorem on the continuity of the sum-function of a power series, which asserts that $(C, 0) \subset (A)$. It has been established that $(C, \alpha) \subset (A)$, for every $\alpha$, however large.

Absolute Abel summability : The series $\Sigma a_n$ is said to be absolutely summable (A), or simply summable $|A|$, if the series $\Sigma a_n x^n$ converges for $0 \leq x < 1$ and its sum-function $f(x)$ is of bounded variation in $[0, 1)$, that is

$$\int_0^1 |f'(x)| \, dx < \infty.$$ [2]

Concerning the relation between convergence and summability $|A|$, Whittaker[3] and Prasad[4] demonstrated that they are independent of each other in that neither is included in the other.

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1) Abel (1), Theorem 4.

2) Whittaker (1), Prasad (1).

3) Whittaker (1).

4) Prasad (1).

$|A|_k$ - summability : The series $\Sigma\, a_n$ is said to be absolutely summable (A) with index k ($k \geq 1$), or simply summable $|A|_k$, if the series $\Sigma\, a_n x^n$ converges for $0 \leq x < 1$ and its sum-function f(x) satisfies the condition

$$\int_0^1 (1-x)^{k-1}\, |f'(x)|^k\, dx < \infty .$$[1)]

Riemann summability : It is well known that a Fourier cosine series

$$\phi\,(t) \sim \sum_{n=1}^{\infty} a_n \cos nt$$

can be integrated termwise, i.e.,

$$\Phi(t) = \int_0^t \phi(u)\, du = \sum_{n=1}^{\infty} n^{-1} a_n \sin nt$$

and the series on the right is uniformly convergent; $\Phi(t)$ is absolutely continuous.[2)] This gives rise to the following definition of summability.

The series $\Sigma\, a_n$ is said to be summable to the sum s, by Lebesgue's method, or simply summable (R, 1), if the sine

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1) Flett (1).

2) Szasz (1).

series

$$(1.6.4) \qquad F(t) = \Sigma n^{-1} a_n \sin nt$$

converges in some interval $0 < t < t_o$, and if

$$(1.6.5) \qquad t^{-1} F(t) \to s, \quad \text{as } t \to 0.$$ [1]

It is known that convergence does not imply Lebesgue summability and conversly, the latter does not imply convergence.[2]

There is an anohter method of summation $(R_1)$ which is similar to that of $(H,1)$.

The series $\Sigma a_n$ is said to be summable $(R_1)$ to the sum s, if the series

$$(1.6.6) \qquad G(t) = \sum_{n=1}^{\infty} a_n n^{-1} \sin nt$$

converges in some interval $0 < t < t_o$, and if

$$(1.6.7) \qquad \frac{2}{\pi} G(t) \to s, \quad \text{as } t \to 0.$$

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1) Zygmund (1), p.27?; Fatou (1).

2) cf. Szász (1).

The methods (R,1) and ($R_1$) are not regular, as the regularity condition $\sum_{n=1}^{\infty} n^{-1} |\sin nt| <$ constant is not satisfied.[1)]

There are two similar but different families of methods of summation of series which stem from the work of Riemann on trigonometric series, which are the generalizations of (R, 1) and ($R_1$)-methods respectively.

<u>(R, p) and ($R_p$)-summability</u>: Let $f_p(x) = \left(\frac{\sin x}{x}\right)^p$, $x \neq 0$ $f_p(0)=1$. A given series $\Sigma\, a_n$ is said to be summable (R,p) (p is a positive integer) to the sum s, if the series

$$(1.6.8) \qquad F_p(t) = \sum_{n=0}^{\infty} a_n f_p(nt)$$

converges in some interval $0 < t < t_0$ and $\lim_{t \to 0} F_p(t) = s$.[2)]

In the special case when p=2, it reduces to the familiar (R, 2)-method which is known as Riemann method of summation.[3)]

The series $\Sigma\, a_n$ is said to be summable ($R_p$) to sum s, if the series

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1) cf. Szasz (1).

2) Verblunsky (1).

3) Riemann (1).

$$G_p(t) = C_p^{-1}\, t \sum_{n=1}^{\infty} a_n f_p(nt), \tag{1.6.9}$$

where

$$C_p = \int_0^{\infty} u^{-p} (\sin u)^p \, du ,$$

converges in some interval $0 < t < t_0$ and $G_p(t) \to 0$ as $t \to 0.$[1] It is well known that $(R, p)$ and $(R_p)$ are regular when $p \geq 2$, but not for $p = 1.$[2]

The methods $(R, 1)$ and $(R_1)$ are special cases of the above methods.

Verblunsky[3] has shown that if $p \geq 2$, then $(R,1) \subseteq (R,p)$, $(R_1) \subseteq (R_p)$ and for $p \geq 3$,[4] $(R, 2) \subseteq (R, p)$, $(R_2) \subseteq (R_p)$ . Kuttner[5] further established that if $p \geq 2$, then $(R,1) \subseteq (R_p)$, $(R_1) \subseteq (R, p)$; and for $p \geq 3$, $(R, 2)$ $(R_p)$, $(R_2) \subseteq (R, p)$.

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1) Verblunsky (1).

2) Hardy (1).

3) Verblunsky (1).

4) see also Kuttner (1).

5) Kuttner (1).

As regard inclusion relation between the methods of summability $(R, p)$ and $(R_p)$ it is known that $(R, p)$ are incomparable,[1] when $p = 1, 2, 3$, even for Fourier series.[2]

Hirokawa gave attention to the analogy between the methods $(R, p)$ and $(R_p)$ and defined the Riemann-Cesaro method which contains these methods as special cases.

$(R, p, \alpha)$-summability : A given series $\Sigma a_n$ is said to be summable by Riemann-Cesaro method of order p and index $\alpha$ (p being a positive integer, and $\alpha$ being a real number not necessarily be an integer) or briefly, summable $(R, p, \alpha)$ to sum s, if the series in

$$(1.6.10) \qquad F_p(\alpha, t) = (C_{p,\alpha})^{-1} t^{\alpha+1} \sum_{n=1}^{\infty} S_n^{\alpha} f_p(nt)$$

where

$$(1.6.11) \quad C_{p,\alpha} = \begin{cases} \dfrac{1}{\Gamma(\alpha+1)} \displaystyle\int_0^{\infty} u^{\alpha-p}(\sin u)^p du & (-1 < \alpha < p-1) ; \\ \pi/2 & (\alpha = 0, \ p = 1) ; \\ 1 & (\alpha = -1), \end{cases}$$

1) Hardy and Rogosinski (1); see also Kuttner (1).

2) cf. Hardy and Rogosinski (1), see also Kuttner (1).

converges in some interval $0 < t < t_0$ and $F_p(\alpha, t) \to s$, as $t \to 0$.[1)]

Under this definition the $(R, p, -1)$ and $(R, p, 0)$-methods are the same as the $(R, p)$ and $(R_p)$-methods respectively. When $p=1$, the method $(R, p, \alpha)$ reduces to $(R, 1, \alpha)$ defined earlier by Hirokawa.[2)] who also established that $(R, 1, \alpha)$-method is not regular.

Hirokawa[3)] has obtained a number of results concerning inclusion relations and other aspects of this method of summability, and also proved that $(R, p, \alpha)$-method is regular for $p \geq 2$.

$(R, \lambda_n, p)$-summability : Let p be a positive real number, and let the sequence $\{\lambda_n\}$ be such that $0 \leq \lambda_0 < \lambda_1 < \ldots < \lambda_n \to \infty$. If the series

$$R^p_\lambda(t) = \sum_{n=0}^{\infty} a_n f_p(\lambda_n t) \tag{1.6.12}$$

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1) Hirokawa (1).

2) Hirokawa (1).

3) Hirokawa (1).

converges in some interval $0 < t < t_o$, and if

$$\lim_{t\to o} R_\lambda^p(t) = s,$$

then we say that the series $\Sigma a_n$ is summable $(R, \lambda_n, p)$ to sum s. [1)]

In the case when $\lambda_n = n$ and p is a positive integer, the summability $(R, \lambda_n, p)$ is the same as the summability (R, p).

(K, 1) and (K, 1, $\alpha$)-<u>summability</u> : Associating with congugate trigonometric series Zygmund [2)] introduced the method (K, 1) which corresponds to the method (R, 1).

A given infinite series $\Sigma a_n$ is said to be summable (K, 1) to sum s, if

$$(1.6.13) \qquad K(t) = a_o + \frac{2}{\pi} \sum_{n=1}^{\infty} a_n \int_t^{\pi} \frac{\sin nx}{2\tan x/2}\, dx$$

converges in some interval $0 < t < t_o$, and $K(t) \to s$, as $t \to 0$.

Zygmund [3)] has proved that (K, 1)-method is not regular

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1) Russell(1). This more general definition has been given by Burkill (1), for p = 1, 2, and by Burkill and Petersen(1) for p rational with odd denominator (which ensures that $f_p(x)$ is real).

2) Zygmund (2).

3) Zygmund (2).

like (B, 1) or ($B_1$)-methods. It has been proved by Izumi[1] that for Fourier series, summability (K, 1) is equivalent to summability ($B_1$).

The method (K, 1) has recently been extended by Hirokawa[2] who introduced a parameter $\alpha$. The idea of generalizing the summability (K, 1) to (K, 1, $\alpha$) is similar to that of summability (B, 1, $\alpha$) defined by him earlier.

The series $\Sigma a_n$ is said to be summable by (K, 1, $\alpha$)-method to sum s ($\alpha$ is a real number such that $-1 \leq \alpha \leq 0$), if

$$(1.6.14) \qquad K(\alpha,t) = B_\alpha^{-1} t^{\alpha+1} \sum_{n=1}^{\infty} S_n^\alpha \int_t^\pi \frac{\sin nx}{2\tan x/2}\, dx$$

converges in some interval $0 < t < t_0$, and if

$$\lim_{t\to 0} K(\alpha,t) = s\,,$$ [3]

where

$$B_\alpha = \begin{cases} \pi/2 & , \ \alpha = -1\,; \\ (\alpha+1)^{-1} \sin(\alpha+1)\pi/2 & , \ -1 < \alpha < 0\,; \\ 1 & , \ \alpha = 0. \end{cases}$$

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1) Izumi (1).
2) Hirokawa (2).
3) Hirokawa (3).

It has been shown by Hirokawa[1] that the method $(K, 1, \alpha)$ is not regular for $-1 < \alpha < 0$. He has also shown that $(K, 1, \alpha)$-method of summation has some properties similar to that of $(R, 1, \alpha)$-method.

<u>Absolute</u> $(R, p)$-<u>summability</u> : Recently Gesberg[2] defined the absolute Riemann summability in the following manner :

The series $\Sigma a_n$ is said to be absolutely summable by Riemann method, or simply, summable $|R, p|$, where p is a positive integer, if the series

$$(1.6.18) \qquad F_p(x) = \sum_{k=1}^{\infty} a_k f_p(kx)$$

is convergent for $x \in (0, \delta)$, for some $\delta > 0$ and $F_p(x) \in BV\,[0, \delta)$ that is,

$$\int_0^{\delta} \left| \frac{d}{dx} (F_p(x) \right| dx < \infty .$$

$|R, p|_k$-<u>summability</u> : We say that the series $\Sigma a_n$ is summable $|R, p|_k$, where p is a positive integer and $k \geq 1$,

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1) Hirokawa (3).

2) Gesberg (1).

if the series (1.6.18) is convergent for $x \in [0, \delta)$, for some $\delta > 0$, and

$$\int_0^{\delta} x^{k-1} \left| \frac{d}{dx} (F_p(x)) \right|^k dx < \infty .$$

The method $|R, p|_1$ is the same as the method $|R, p|$. For $k > 1$ the methods $|R, p|$ and $|R, p|_k$ are independent.

1.7 Concerning the relation between Cesàro summability and Riemann summability, Hardy and Littlewood [1] proved the following :

Theorem A. If a series $\Sigma a_n$ is summable $(C, -\alpha)$ for some $\alpha > 0$ then it implies summability $(R, 1)$ of the series $\Sigma a_n$ to the same sum.

Replacing $(C, -\alpha)$ summability by a more general assumption Szász [2] proved the following theorem :

Theorem B. If $\Sigma a_n$ is summable $(C, 1-\alpha)$ for some positive $\alpha < 1$, and, if

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1) Hardy and Littlewood (1).

2) Szász (2), (3).

(1.7.1) $$\sum_{\nu=1}^{n} |S_\nu^{-\alpha}| = O(n^{1-\alpha}) \quad \text{as } n \to \infty,$$

then the series $\Sigma a_n$ is summable by Lebesgue and $(R_1)$-methods.

This theorem was subsequently generalized by Varshney[1] in the sense that he replaced $(C, 1-\alpha)$ summability by harmonic summability :

Theorem C.[2] If a series $\Sigma a_n$ is harmonic summable and if

(1.7.2) $$\sum_{\nu=1}^{n} |H_\nu - H_{\nu-1}| = O(\log n) \quad \text{as } n \to \infty,$$

then the series $\Sigma a_n$ is Lebesgue summable.

Another similar generalization of Theorem A of Szasz (in the direction of $(R_1)$-summability) is due to Singh[3] who has shown :

Theorem D. If a series $\Sigma a_n$ is harmonic summable and

(1.7.3) $$\sum_{\nu=1}^{n} |H_\nu - H_{\nu-1}| = O(\log n), \quad \text{as } n \to \infty,$$

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1) Varshney (2)

2) Throughout we use $H_n$ to denote n-th harmonic sum of the series $\Sigma a_n$.

3) Singh (1).

then the series $\Sigma\, a_n$ is summable $(R_1)$.

With a view to generalize Theorems A-D, we have established the following theorems in Chapter II.

*Theorem 1. If* $\Sigma\, a_n$ *is* $(N,\ p_n)$ *summable and, if*

$$\sigma_n = \sum_{v=1}^{n} \left| T_v - T_{v-1} \right| = O(P_n),$$

*then* $\Sigma\, a_n$ *is* $(R,\ 1)$*-summable, provided that* $\{p_n\}$ *is non-negative, non-increasing sequence such that* $P_n \to \infty$, *and*

$$\text{(i)} \quad d_n = \sum_{v=0}^{n} c_v = O\left(\frac{1}{P_n}\right);$$

$$\text{(ii)} \quad \sum_{v=n+1}^{\infty} |c_v| = O\left(\frac{1}{P_n}\right), \quad \text{for } n \geq 0;$$

$$\text{(iii)} \quad \sum_{v=n}^{\infty} \frac{P_{v-n}}{v(v+1)} = O\left(\frac{P_n}{n}\right);$$

$$\text{(iv)} \quad \sum_{v=0}^{n} \frac{1}{P_v} = O\left(\frac{n}{P_n}\right);$$

(v) *for a positive integer* $\mu$ *and* $n = [\mu t^{-1}]$, $\tau = [t^{-1}]$

$$P_n = O(P_\mu P_\tau).$$

<u>Theorem 2. Under the hypotheses of Theorem 1,</u> $\Sigma a_n$ <u>is summable</u> $(R_1)$.

From these theorems we have also deduced (with the help of a lemma of Varshney [1]) a couple of corollaries which are easy to apply.

In Chapter III we have obtained as Theorem 1 a generalization of Theorems 1 and 2 of Chapter II for summability $(R, 1, \alpha)$, in the sense that these theorems become special cases of the theorem proved therein. Theorem 2 of Chapter III gives a generalization of corollaries I and II and hence it also generalizes the results of Czasz, Varshney and Singh mentioned earlier.

Since the methods $(R, 1, \alpha)$ and $(K, 1, \alpha)$ have some properties in common, in view of Theorems 1 and 2 of Chapter III, the question arises : Whether it is possible to prove these theorems for summability $(K, 1, \alpha)$. In Chapter IV, we answer this question in the affirmative by proving an analogous result for summability $(K, 1, \alpha)$.

Regarding the total regularity of $(R, 2)$-method Lee [2]

1) Varshney (1), Lemma 2.

2) Lee (1).

proved the following theorem :

Theorem E. If $a_n \to 0$ and $a_n > -1/n$ , then the $(R, 2)$-method is totally regular.

Later, in 1965, Hirokawa [1] generalised this result for $(R, p)$-summability. His result is the following :

Theorem F. Let $p=1, 2, \ldots$ , Suppose that $a_n \geq -K/n$ $(n=1, 2, 3, \ldots,$ $K$ ; a positive constant),

$$\sum_{n=1}^{\infty} a_n = +\infty$$

and

$$\sum_{n=1}^{\infty} a_n f_p(nt)$$

converges in $0 < t < t_0$. Then

$$\lim_{t\to 0} \sum_{n=1}^{\infty} a_n f_p(nt) = +\infty.$$

Concerning the summability methods $(R_p)$ and $(R, p, \alpha)$ Hirokawa [2] also established the following theorems.

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1) Hirokawa (4).

2) Hirokawa (4).

Theorem G. The method $(R_{2p})$, $p=1,2,\ldots$ is totally regular.

Theorem H. The method $(R_{2p+1})$, $p=1,2,\ldots$ is not totally regular.

Theorem I. The method $(R, 2p, \alpha)$, $p=1,2,\ldots$, $0 \leq \alpha < 2p$ is totally regular.

We have also studied the total regularity of the summability methods $(R, p, \alpha)$ and $(R, \lambda_n, p)$ in Chapter V of the present Thesis and have established a couple of theorems which generalise some of the above mentioned results. We prove

<u>Theorem 1. The method</u> $(R, 2p+1, \alpha)$, $0 \leq \alpha < 2p$, $p \geq 1$ <u>is not totally regular.</u>

<u>Theorem 2. Let</u> $p=1,2,\ldots$ <u>and</u> $\Lambda_n = \lambda_{n+1}/(\lambda_{n+1} - \lambda_n)$, $0 \leq \lambda_0 < \lambda_1 < \ldots < \lambda_n \to \infty$, <u>such that</u> $\sum_{n=\nu}^{\infty} \frac{1}{\Lambda_n \lambda_n^p} = \left(\frac{1}{\lambda_\nu^p}\right)$, <u>and, for every sequence</u> $\{t_\nu\}$ <u>of positive numbers tending to zero and integers</u> $N_\nu$; $\lim_{\nu\to\infty} \lambda_{N_\nu} t_\nu = x$. <u>If</u> $a_n \geq -K/\Lambda_n$ $(n=1,2,\ldots,$ $K$; <u>positive constant</u>), $\sum_{n=1}^{\infty} a_n f_p(\lambda_n t)$ <u>converges in</u> $0 < t < t_0$, <u>then</u> $\sum a_n = +\infty$ <u>implies</u>

$$\lim_{t\to 0} \sum_{n=1}^{\infty} a_n f_p(\lambda_n t) = +\infty.$$

Concerning the relation between absolute Abel and absolute Riemann summability methods, Gesberg [1] has recently established the following theorem.

Theorem J. If the series $\Sigma a_n$ is summable $|R, p|$ and, if, $a_n = O(1)$, then it is also summable $|A|$.

In the sixth and the last chapter of the present Thesis, we extend Theorem J for the methods $|A|_k$ and $|R, p|_k$, since it is known that, for $k > 1$, the methods $|A|$ and $|R, p|$ are independent of the methods $|A|_k$ and $|R, p|_k$ respectively. We prove

Theorem 1. Let $k \geq 1$. If the series $\Sigma a_n$ is summable $|R, p|_k$ and, if $a_n = O(1)$, then it is also summable $|A|_k$.

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1) Gesberg (1).

## Chapter II

## ON THE RELATION OF NÖRLUND SUMMABILITY WITH LEBESGUE AND $(R_1)$-SUMMABILITY

2.1. Definitions and Notations : Let $\Sigma a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$ . Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + \dots + p_n \neq 0, \ P_{-1} = p_{-1} = 0.$$

The sequence-to-sequence transformation :

$$(2.1.1) \qquad t_n = \frac{T_n}{P_n} = \sum_{v=0}^{n} \frac{p_{n-v}\, s_v}{P_n}$$

defines the sequence $\{t_n\}$ of Nörlund means[1] of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$ .

The series $\Sigma a_n$ is said to be summable $(N, p_n)$ to sum $s$, if $\lim_{n\to\infty} t_n$ exists and is equal to $s$ .

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1) Nörlund (1) , Woronoi (1).

In the special case in which

$$p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)} \quad (\alpha \geq 0) , \tag{2.1.2}$$

the Nörlund mean reduces to the familiar $(C, \alpha)$-mean [1]. The summability $(N, p_n)$, with $p_n$ defined by (2.1.2), is thus the same as summability $(C,\alpha)$ [2].

Similarly, in the case in which

$$\begin{cases} p_n = 1/(n+1) & (n \geq 0), \\ P_n = 1+ (1/2)+\ldots+(1/(n+1)) \sim \log n, \text{ as } n\to\infty, \end{cases} \tag{2.1.3}$$

the Nörlund mean reduces to the familiar harmonic mean [3] and summability $(N, p_n)$ is then the same as harmonic summability , or simply the summability $(N, 1/(n+1))$. It is known that the summability $(N, 1/(n+1))$ implies summability $(C, \alpha)$ for every $\alpha > 0$.

The necessary and sufficient conditions for the regularity of the method of summability $(N, p_n)$, defined by (2.1.1), are :

$$\lim_{n\to\infty} \frac{p_n}{P_n} = 0 , \tag{2.1.4}$$

---

1) Hardy (1), § 5.13.

2) Hardy (1), § 5.13.

3) Hardy (1), § 5.13.

and

(2.1.5) $$\sum_{\nu=0}^{n} |p_\nu| = O(P_n) \text{ , as } n\to\infty .$$

If $p_n$ is real and non-negative (2.1.5) is automatically satisfied and then (2.1.4) is the only necessary and sufficient condition for the regularity of the method $(N, p_n)$. If, in addition, $p_n$ is non-increasing, then condition (2.1.4) is also satisfied. Thus the summability $(N, (n+1)^{-1})$ is regular .

We observe that the regularity condition (2.1.4) implies that

$$\lim_{n\to\infty} \frac{P_{n-1}}{P_n} = \lim_{n\to\infty} (1 - \frac{p_n}{P_n}) = 1,$$

and

$$\lim_{n\to\infty} \frac{P_n}{P_{n-1}} = \lim_{n\to\infty} 1/(1-\frac{p_n}{P_n}) = 1.$$

The series $\Sigma\, a_n$ is said to be summable by Lebesgue method (shortly (R,1)-summable) to the sum s, if the sine series

(2.1.6) $$F(t) = \sum_{n=1}^{\infty} a_n \left(\frac{\sin nt}{n}\right)$$

is convergent in some interval $-\tau < t < \tau$, and

(2.1.7) $$t^{-1} F(t) \to s\,, \text{ as } t \to 0.$$ [1]

The series $\Sigma\, a_n$ is said to be summable by $(R_1)$-method to the sum $s$ , if the series

(2.1.8) $$G(t) = \sum_{n=1}^{\infty} a_n \left(\frac{\sin nt}{n}\right)$$

converges in some interval $-\tau < t < \tau$, and if

(2.1.9) $$\frac{2}{\pi}\, G(t) \to s\,, \text{ as } t \to 0\,.$$ [2]

It is convenient to assume $a_0 = 0$.

The methods $(R, 1)$ and $(R_1)$ are not regular as the regularity condition $\sum_{n=1}^{\infty} n^{-1} |\sin nt| <$ constant is not satisfied, [3] and it is also known that these methods are not comparable even for Fourier series. [4]

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1) Fatou (1).

2) Riemann (1).

3) cf. Szasz (1).

4) Hardy and Rogosinski ( 1 ).

We write $H_n$ to denote the n-th harmonic sum of the sequence $\{a_n\}$.

2.2. We set

(2.2.1) $$\left(\sum_{n=0}^{\infty} p_n x^n\right)^{-1} = \sum_{n=0}^{\infty} c_n x^n, \quad c_0=1, \quad (|x|<1),$$

so that from (2.1.1),

2.1.1 $$\sum_{n=0}^{\infty} T_n x^n = \sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} P_n x^n$$

$$= \left(\sum_{n=0}^{\infty} a_n x^n\right)(1-x)^{-1}\left(\sum_{n=0}^{\infty} p_n x^n\right),$$

and therefore,

(2.2.2) $$\sum_{n=0}^{\infty} a_n x^n = (1-x)\left(\sum_{n=0}^{\infty} T_n x^n\right)\left(\sum_{n=0}^{\infty} c_n x^n\right)$$

$$= \left(\sum_{n=0}^{\infty} (T_n - T_{n-1}) x^n\right) \sum_{n=0}^{\infty} c_n x^n )$$

and

(2.2.3) $$\sum_{n=0}^{\infty} s_n x^n = \left(\sum_{n=0}^{\infty} T_n x^n\right) . \left(\sum_{n=0}^{\infty} c_n x^n\right),$$

from which we obtain

(2.2.4) $$s_n = \sum_{v=0}^{n} c_{n-v} T_v ,$$

and

(2.2.5) $$a_n = \sum_{v=0}^{n} c_{n-v} (T_v - T_{v-1}).$$

In what follows we take $a_0 = 0$, so that $T_0 = 0$.

2.3 Introduction. Concerning Lebesgue summability and $(R_1)$-summability Szász [1] has proved the following results.

Theorem A. If $\Sigma a_n$ is summable $(C, 1-\alpha)$ for some positive $\alpha < 1$, and if

$$(2.3.1) \qquad \sum_{v=1}^{n} \left|S_v^{-\alpha}\right| = O(n^{1-\alpha}), \text{ as } n \to \infty,$$

then the series $\Sigma a_n$ is summable by Lebesgue, $(R, 1)$ and $(R_1)$-methods.

Recently, in the case of Lebesgue summability Varshney [2] has proved the following analogous theorem for harmonic summability :

Theorem B. If a series $\Sigma a_n$ is harmonic summable and, if

$$(2.3.2) \qquad \sum_{v=1}^{n} \left|H_v - H_{v-1}\right| = O(\log n),$$

then $\Sigma a_n$ is Lebesgue summable.

1) Szász (2), (3).

2) Varshney (2).

Another generalisation of the Theorem A of Szasz (in the direction of $(H_1)$-summability) is due to Singh [1)] who has proved:

Theorem C. If a series $\Sigma a_n$ is harmonic summable and

$$(2.3.3) \qquad \sigma_n = \sum_{v=1}^{n} \left| H_v - H_{v-1} \right| = O(\log n), \text{ as } n\to\infty ,$$

then, the series $\Sigma a_n$ is summable $(H_1)$.

In this chapter we prove two Theorems. Theorem 1 covers both the Theorem B and the first part of Theorem A as special cases, and Theorem 2 covers the Theorem C and the second part of Theorem A.

In the end we deduce two interesting corollaries from our Theorems, which are more general than the above-mentioned theorems and are also easy to apply.

2.4 We establish the following theorems.

<u>Theorem</u> 1. <u>If</u> $\Sigma a_n$ <u>is</u> $(N, p_n)$-<u>summable and</u>, <u>if</u>

$$\sigma_n = \sum_{v=1}^{n} \left| t_v - t_{v-1} \right| = O(P_n),$$

---

1) Singh (1).

*then* $\Sigma a_n$ *is* $(R,1)$-*summable*, *provided* $\{p_n\}$ *is non-negative, non-increasing sequence such that* $P_n \to \infty$, *and*

$$\text{(i)} \quad d_n = \sum_{v=0}^{n} c_v = O\left(\frac{1}{P_n}\right);$$

$$\text{(ii)} \quad \sum_{v=n+1}^{\infty} |c_v| = O\left(\frac{1}{P_n}\right), \text{ for } n \geq 0;$$

$$\text{(iii)} \quad \sum_{v=n}^{\infty} \frac{P_{v-n}}{v(v+1)} = O\left(\frac{P_n}{n}\right);$$

$$\text{(iv)} \quad \sum_{v=0}^{n} \frac{1}{P_v} = O\left(\frac{n}{P_n}\right);$$

(v) *for a positive integer* $\mu$ *and* $n = [\mu t^{-1}]$, $\tau = [t^{-1}]$

$$P_n = O(P_\mu P_\tau).$$

*Theorem* 2. *Under the hypotheses of Theorem* 1, $\Sigma a_n$ *is summable* $(R_1)$.

2.5 We need the following lemmas for the proof of our theorems.

*Lemma* 1. *If* $p_n$ *is a non-negative, non-increasing sequence such that the series*

$$\sum_{v=n}^{\infty} \frac{P_{v-n}}{v(v+1)}$$

*converges, then* $P_n/n \to 0$, *as* $n \to \infty$.

<u>Proof.</u> Since $p_n \geq 0$ and $np_n \leq P_n$, we have

$$\frac{P_n}{n} - \frac{P_{n+1}}{n+1} = \frac{P_n - np_{n+1}}{n(n+1)} \geq \frac{P_n - np_n}{n(n+1)} \geq 0.$$

Obviously, $\frac{P_n}{n} > 0$. Thus the sequence $\{P_n/n\}$ is bounded and non-increasing. Hence, there exists $\lim\limits_{n\to\infty} \frac{P_n}{n} = \alpha$, say. Then there exists an integer $N$ such that $\frac{P_n}{n} > \alpha/2$ $(= \alpha - \alpha/2)$ for $n \geq N$.

Hence, we have, for $n \geq N$,

$$\sum_{v=n}^{\infty} \frac{P_{v-n}}{v(v+1)} \geq \sum_{v=2n}^{\infty} \frac{P_{v-n}}{v-n} \cdot \frac{v-n}{v(v+1)}$$

$$\geq \alpha/2 \sum_{v=2n}^{\infty} \frac{v-n}{v(v+1)} = \infty,$$

which contradicts our assumption that the series $\sum\limits_{v=n}^{\infty} \frac{P_{v-n}}{v(v+1)}$ converges. Thus we see that $\alpha = 0$ and hence the result.

<u>Lemma 2. Let $\{p_n\}$ be a non-negative, non-increasing sequence such that</u>

$$\sum_{v=n}^{\infty} \frac{P_{v-n}}{v(v+1)} = O\left(\frac{P_n}{n}\right).$$

 

Then, for $n \geq 1$,

$$\sum_{v=n}^{\infty} \frac{P_v}{v(v+1)} = O\left(\frac{P_n}{n}\right) . \tag{2.8.1}$$

Proof. We have

$$\sum_{v=n}^{\infty} \frac{P_v}{v(v+1)} = \sum_{v=n}^{\infty} \frac{P_v - P_{v-n}}{v(v+1)} + \sum_{v=n}^{\infty} \frac{P_{v-n}}{v(v+1)}$$

$$= \sum_{v=n}^{2n} \frac{(P_v - P_{v-n})}{v(v+1)} + \sum_{v=2n+1}^{\infty} \frac{(P_v - P_{v-n})}{v(v+1)} + \sum_{v=n}^{\infty} \frac{P_{v-n}}{v(v+1)}$$

$$= O\left(\frac{P_{2n}}{n}\right) + \sum_{v=2n+1}^{\infty} \frac{1}{v(v+1)} \sum_{\mu=v-n+1}^{v} p_\mu + O\left(\frac{P_n}{n}\right),$$

$$= O\left(\frac{P_n}{n}\right) + O\left(\frac{P_n}{n}\right) + O\left(np_n \sum_{v=2n+1}^{\infty} \frac{1}{v(v+1)}\right)$$

$$= O\left(\frac{P_n}{n}\right),$$

by hypothesis, since $(n+1)\, p_n \leq P_n$ .

This proves Lemma 2.

<u>Lemma 3. Let</u> $p_n$ <u>be non-negative, non-increasing such</u> $\left\{\frac{P_n}{n}\right\}$ <u>is a null sequence. If</u> $\Sigma\, a_n$ <u>is summable</u> $(N, p_n)$, <u>then</u>

$$\text{(i)}\quad W_n = \sum_{v=n}^{\infty} \frac{T_v - T_{v-1}}{v} = o\left(\frac{P_n}{n}\right);$$

$$\text{(ii)}\quad W_n' = \sum_{v=1}^{n} W_v = o\,(P_n).$$

<u>Proof</u>. (i) We may assume without loss of generality, that $T_n = o(P_n)$.

By Abel's transformation and by hypothesis, as $m \to \infty$, we have

$$W_n = \sum_{v=n}^{m} \frac{T_v - T_{v-1}}{v} = \sum_{v=n}^{m-1} \frac{1}{v(v+1)} \sum_{\mu=0}^{v} (T_\mu - T_{\mu-1}) +$$

$$+ \frac{1}{m} \sum_{\mu=0}^{m} (T_\mu - T_{\mu-1}) - \frac{1}{n} \sum_{\mu=0}^{n-1} (T_\mu - T_{\mu-1})$$

$$= \sum_{v=n}^{m-1} \frac{T_v}{v(v+1)} + \frac{T_m}{m} - \frac{T_{n-1}}{n}$$

$$= o\left(\sum_{v=n}^{\infty} \frac{P_v}{v(v+1)}\right) + o\left(\frac{P_{n-1}}{n}\right)$$

$$= o\left(\frac{P_n}{n}\right) + o\left(\frac{P_{n-1}}{n}\right) \quad \text{( by Lemma 2).}$$

$$= o\left(\frac{P_n}{n}\right),$$

by regularity of the method $(N, p_n)$.

$$\text{(ii)}\quad W_n' = \sum_{v=1}^{n} W_v = \sum_{v=1}^{n} v \,\Delta W_v + n\, W_{n+1}$$

$$= \sum_{v=1}^{n} v\, \frac{(T_v - T_{v-1})}{v} + n\, W_{n+1}$$

$$= \sum_{v=1}^{n} (T_v - T_{v-1}) + n\, W_{n+1}$$

$$= T_n + n\, W_{n+1} \quad (\text{ since } T_0 = p_0\, a_0 = 0)$$

$$= O(P_n) + O\left(n.\frac{P_{n+1}}{n+1}\right)$$

$$= O(P_n),$$

by hypothesis and (i). Hence the result.

<u>Lemma 4.</u>[1] <u>If</u> $p(x) = \Sigma\, p_n x^n$ <u>is convergent for</u> $|x| < 1$, <u>and</u>

$$\text{(2.5.2)}\qquad p_0 = 1,\ p_n > 0,\ \frac{p_{n+1}}{p_n} \geq \frac{p_n}{p_{n-1}} \quad (n > 0),$$

---

1) Harday (1), Theorem 22. This is proved with a different purpose, by Szegö(1), who attributes the result to Kalusa.

*then*

$$(2.5.3) \qquad \{p(x)\}^{-1} = 1 + c_1 x + c_2 x^2 + \dots$$

*where* $c_n \leq 0$, *for* $n=1, 2, \dots$, $\sum_{n=1}^{\infty} |c_n| \leq 1$. *If* $\Sigma p_n = \infty$,

*then* $\sum_{n=1}^{\infty} |c_n| = 1$.

*Lemma 5.*[1] *If* $\{p_n\}$ *is a non-negative and non-increasing sequence such that* $p_0 = 1$, $P_n \to \infty$, *and* $\{p_{n+1}/p_n\}$ *is non-decreasing sequence, then*, for $n > 0$,

$$d_n = \sum_{v=n+1}^{\infty} |c_v| = \sum_{v=0}^{n} c_v = O\left(\frac{1}{P_n}\right).$$

*Remark.* The identity

$$d_n = \sum_{v=n+1}^{\infty} |c_v| = \sum_{v=0}^{n} c_v$$

is obtained by virtue of the Lemma 4.

*Lemma 6.*[2] *Let* $\Delta_n^m \phi(nt)$ *denote the m-th difference of*

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1) Varshney (1), Lemma 2.

2) Obrechkoff (1).

$\phi(nt)$ with respect to $n$. Then, we have

$$(2.5.4) \qquad \Delta_n^m \, \phi(nt) = O\left( t^{m-p}/n^p \right),$$

where $m$ is a non-negative integer and $\phi(t) = \left( \frac{\sin t}{t} \right)^p$.

Lemma 7. If $\{p_n\}$ is such that it satisfies all the conditions of Theorem 1 except (iii), then the series

$$(2.5.5) \qquad \sum_{n=0}^{\infty} c_n \frac{\sin(n+v)t}{(n+v)t} = S_v(t)$$

is absolutely convergent and, for $m=0,1,2,\ldots$,

$$(2.5.6) \qquad \Delta_v^m \, S_v(t) = O\left( \frac{t^{m-1}}{vP_\tau} \right).$$

Absolute convergence of the series (2.5.5) follows from the hypotheses, since $\sum_{n=1}^{\infty} |c_n| < \infty$. To prove (2.5.6) we have, by setting

$$\psi(t) = \left( \frac{\sin t}{t} \right),$$

$$\Delta_v^m \, S_v(t) = \Delta_v^m \left\{ \sum_{n=0}^{\infty} c_n \, \psi((n+v)t) \right\}$$

$$= \sum_{n=0}^{\infty} c_n \, \Delta_v^m \, \psi((n+v)t)$$

$$= \left( \sum_{n=0}^{\tau} + \sum_{n=\tau+1}^{\infty} \right) c_n \Delta_v^m \psi((n+v)t),$$

$$= S_v^{(1)}(t) + S_v^{(2)}(t) \text{ , say .}$$

Now, by hypothesis and Lemma 5, we have

$$S_v^{(2)}(t) = \sum_{n=\tau+1}^{\infty} c_n \Delta_v^m \psi((n+v)t)$$

$$= O\left[ \sum_{n=\tau+1}^{\infty} |c_n| \cdot \frac{t^{m-1}}{(n+v)} \right]$$

$$= O\left( \frac{t^{m-1}}{v+\tau+1} \sum_{v=\tau+1}^{\infty} |c_n| \right)$$

$$= O\left( \frac{t^{m-1}}{vP_\tau} \right).$$

And, on applying Abel's transformation to the expression in $S_v^{(1)}(t)$, and by hypotheses (i) and (iv) and Lemma 6, we have

$$S_v^{(1)}(t) = \sum_{n=0}^{\tau-1} d_n \Delta_n \left\{ \Delta_v^m \psi((n+v)t) \right\}$$

$$+ d_\tau \Delta_v^m \psi((\tau+v)t),$$

$$= \sum_{n=0}^{\tau-1} d_n \Delta_n^{m+1} \psi((n+v)t) + d_\tau \Delta_v^m \psi((\tau+v)t),$$

$$= O\left( \sum_{n=0}^{\tau-1} \frac{1}{P_n} \frac{t^m}{(n+v)} \right) + O\left( \frac{1}{P_\tau} \frac{t^{m-1}}{(v+\tau)} \right)$$

$$= O\left( \frac{t^m}{v} \sum_{n=0}^{\tau-1} \frac{1}{P_n} \right) + O\left( \frac{t^{m-1}}{v\, P_\tau} \right)$$

$$= O\left( \frac{t^{m-1}}{v\, P_\tau} \right).$$

This completes the proof of Lemma 7.

<u>Lemma 8.</u>[1] <u>If the series</u> (2.1.8) <u>converges</u> in $0 < t < t_0$, <u>then</u>

$$\sum_{n=1}^{\infty} a_n \frac{\sin nt}{n} = \sum_{n=1}^{\infty} a_n \,\rho_n(t)\,, \tag{2.5.7}$$

<u>where</u>

$$\rho_n(t) = \sum_{v=n}^{\infty} \frac{\sin vt}{v}\,. \tag{2.5.8}$$

<u>Conversely, if</u> $n^{-1}a_n \to 0$, <u>then convergence of the right-hand side of</u> (2.5.8) <u>implies</u> (2.5.7).

<u>Lemma 9. If</u> $\{p_n\}$ <u>is such that it satisfies all the conditions of the Theorem 1 except</u> (iii), <u>then the series</u>

---

1) Szász (3).

$$(2.5.9)\qquad \sum_{n=0}^{\infty} c_n \; \rho_{n+v}(t) = B_v(t),$$

where

$$\rho_n(t) = \sum_{v=n}^{\infty} \frac{\sin vt}{v},$$

<u>is absolutely convergent and, for</u> $m = 0,1,2, \ldots$,

$$(2.5.10)\qquad \Delta_v^m B_v(t) = O\left(\frac{1}{v}\frac{n-1}{P_\tau}\right).$$

The proof of this lemma is similar to that of Lemma 7.

2.6 <u>Proof of Theorem 1</u>. Employing (2.1.1) and (2.2.5), we have, by assuming without loss of generality, $T_n = o(P_n)$, as $n\to\infty$,

$$F(t) = \sum_{n=1}^{\infty} n^{-1} \sin nt \sum_{v=1}^{n} c_{n-v} (T_v - T_{v-1})$$

$$= t \sum_{v=1}^{\infty} (T_v - T_{v-1}) \sum_{n=v}^{\infty} c_{n-v} \frac{\sin nt}{nt},$$

the interchange of order of summations being legitimate, since by the following considerations the double series is absolutely convergent.

Since by hypotheses $\Sigma |c_n| < \infty$, we have

$$\left|\sum_{n=v}^{\infty} n^{-1} c_{n-v} \sin nt\right| \leq \frac{1}{v} \sum_{n=0}^{\infty} |c_n| = O(1/v),$$

and hence as $m \to \infty$,

$$\sum_{v=1}^{m} \left|(T_v - T_{v-1})\right| \left| \sum_{n=v}^{\infty} c_{n-v} \frac{\sin nt}{n} \right|$$

$$= O(1) \sum_{v=1}^{m} \frac{|T_v - T_{v-1}|}{v}$$

$$= O(1)(m^{-1} \sigma_m) + O(1) \left( \sum_{v=1}^{m-1} \sigma_v \frac{1}{v(v+1)} \right),$$

$$= O(1) \frac{P_m}{m} + O(1) \sum_{v=1}^{m-1} \frac{P_v}{v(v+1)},$$

$$= O(1),$$

by Lemmas 1 and 2.

Thus

$$\frac{F(t)}{t} = \sum_{v=1}^{\infty} (T_v - T_{v-1}) \sum_{n=v}^{\infty} n^{-1} c_{n-v} \frac{\sin nt}{t},$$

$$= \sum_{v=1}^{\infty} (T_v - T_{v-1}) S_v(t),$$

$$= \left( \sum_{v=1}^{n} + \sum_{v=n+1}^{\infty} \right) (T_v - T_{v-1}) S_v(t),$$

(2.6.1) $$= \Sigma_1 + \Sigma_2, \text{ say}.$$

Now,

$$|\Sigma_2| = \left|\sum_{v=n+1}^{\infty} (T_v - T_{v-1})\, S_v(t)\right|$$

$$= O\left(\sum_{v=n+1}^{\infty} |T_v - T_{v-1}| \frac{1}{vt\, P_\tau}\right)$$

$$= O\left[\frac{1}{t\, P_\tau}\left(\sum_{v=n+1}^{\infty} \frac{\sigma_v}{v(v+1)} - \frac{\sigma_n}{n+1}\right)\right]$$

$$= O\left[\frac{\tau}{P_\tau}\left(\sum_{v=n+1}^{\infty} \frac{P_v}{v(v+1)} + \frac{P_n}{n+1}\right)\right]$$

$$= O\left[\frac{\tau}{P_\tau}\;\frac{P_n}{n}\right] = \quad \left(\frac{P_\mu}{\mu}\right)$$

$$= O(1)\,\frac{P_\mu}{\mu}, \tag{2.6.2}$$

by hypothesis and Lemmas 2 and 7.

Next, we have

$$\Sigma_1 = \sum_{v=1}^{n} (T_v - T_{v-1})\, S_v(t)$$

$$= \sum_{v=1}^{n} (W_v - W_{v+1})\, v\, S_v(t)$$

$$= \sum_{v=1}^{n} W_v \left[v\, S_v(t) - (v-1) S_{v-1}(t)\right] - n\, W_{n+1} S_n(t)$$

$$= - \sum_{v=1}^{n} W_v \, v \left[ S_{v-1}(t) - S_v(t) \right]$$

$$+ \sum_{v=1}^{n} W_v \, S_{v-1}(t) - n \, W_{n+1} \, S_n(t)$$

$$= - \Sigma_{1,1} + \Sigma_{1,2} - n \, W_{n+1} \, S_n(t) \, ,$$

where, by Lemmas 1, 3(ii) and 7 ,

$$\Sigma_{1,1} = \sum_{v=1}^{n} v \, W_v \, \Delta S_{v-1}(t)$$

$$= \sum_{v=1}^{n} \left\{ \sum_{\mu=1}^{v} \mu \, W_\mu \right\} \Delta^2 S_{v-1}(t) + \Delta S_n(t) \sum_{v=1}^{n} v \, W_v \, ,$$

$$= O\left( \sum_{v=1}^{n} v \, P_v \frac{t}{v \, P_\tau} \right) + O\left( \frac{1}{n \, P_\tau} \cdot n \, P_n \right)$$

$$= O\left( nt \frac{P_n}{P_\tau} \right) + O\left( \frac{P_n}{P_\tau} \right)$$

$$= O(\mu P_\mu) + O(P_\mu)$$

$$= O(1) \, ,$$

Since $\sum_{v=1}^{n} v \, W_v = O\left( \sum_{v=1}^{n} v \frac{P_v}{v} \right) = O(n \, P_n)$, and by applying Abel's transformation twice, writing $W'_m = \sum_{\mu=1}^{m} W_\mu$ and by virtue of Lemmas 1, 3(ii) and 7 , we have

$$\Sigma_{1,2} = \sum_{v=1}^{n} \Big( \sum_{m=1}^{v} W'_m \Big) \Delta^2 S_{v-1}(t) + \Delta S_n(t) \sum_{v=1}^{n} W'_v + S_n(t) W'_n$$

$$= O\Big( \sum_{v=1}^{n} v\, P_v \frac{t}{vP_\tau} \Big) + O\Big(\frac{1}{nP_\tau} \sum_{v=1}^{n} P_v\Big) + O\Big(\frac{P_n}{nt\, P_\tau}\Big)$$

$$= O\Big(\frac{nt\, P_n}{P_\tau}\Big) + O\Big(\frac{P_n}{P_\tau}\Big) + O\Big(\frac{P_\mu}{\mu}\Big)$$

$$= O(1)\, \mu\, P_\mu + O(1)\, P_\mu + O(1)\, \frac{P_\mu}{\mu}$$

$$= O(1) \,;$$

and, by Lemmas 1, 3(i) and 7, we have $n\, W_{n+1}\, S_n(t) = O(1)\frac{P_\mu}{\mu}$ $= O(1)$.

Hence

$$\Sigma_1 = O(1) \,. \tag{2.6.3}$$

Therefore, from (2.6.1), (2.6.2) and (2.6.3) we obtain

$$t^{-1} F(t) = O(1) \frac{P_\mu}{\mu} + o(1) \,, \text{ as } t \to 0.$$

Consequently,

$$\limsup_{t \to +0} t^{-1} \left| F(t) \right| \le O(1) \frac{P_\mu}{\mu} \,.$$

$\mu$ being arbitrary large and $O(1)$ independent of $\mu$, we finally get

$$t^{-1} F(t) \to 0 \quad \text{as} \quad t \to 0 .$$

This completes the proof of Theorem 1.

2.7. Proof of Theorem 2. As in the proof of Theorem 1, we may assume, without any loss in generality, that $T_n = O(P_n)$, as $n \to \infty$. Since

$$\frac{a_n}{n} = \frac{1}{n} \sum_{v=1}^{n} c_{n-v} T_v = O(\frac{P_n}{n}) \sum_{v=1}^{n} | c_{n-v} | = O(1),$$

by hypotheses (ii) and (iii) and Lemma 1, therefore, by virtue of Lemma 8, it is sufficient to prove that $\sum_{1}^{\infty} a_n \rho_n(t)$ converges in the interval $-\tau < t < \tau$ and its limit, as $t \to 0$, is zero.

Employing (2.2.5), we have

$$\sum a_n \rho_n(t) = \sum_{n=1}^{\infty} \rho_n(t) \sum_{v=1}^{n} c_{n-v} (T_v - T_{v-1})$$

$$= \sum_{v=1}^{\infty} (T_v - T_{v-1}) \sum_{n=v}^{\infty} c_{n-v} \rho_n(t) ,$$

the interchange of order of summations being legitimate, since by the following considerations the double series is absolutely convergent.

Since by hypothesis (ii), $\sum_{n=0}^{\infty} |c_n| < \infty$, and by the fact that

$$\rho_n(t) = O(\frac{1}{nt})^{1)},$$

for every fixed $t > 0$, we have

$$\sum_{v=1}^{\infty} |T_v - T_{v-1}| \sum_{n=0}^{\infty} |c_n \rho_{n+v}(t)|$$

$$= O\left( \sum_{v=1}^{\infty} |T_v - T_{v-1}| \frac{1}{vt} \sum_{n=0}^{\infty} |c_n| \right)$$

$$= O\left( \sum_{v=1}^{\infty} \frac{1}{v} |T_v - T_{v-1}| \right).$$

Now, as $m \to \infty$,

$$\sum_{v=1}^{m} \frac{1}{v} |T_v - T_{v-1}| = O(m^{-1} \sigma_m) + O\left( \sum_{v=1}^{m-1} \frac{\sigma_v}{v(v+1)} \right)$$

$$= O(1) \frac{P_m}{m} + O(1) \sum_{v=1}^{m-1} \frac{P_v}{v(v+1)}$$

$$= O(1),$$

by hypothesis and Lemmas 1 and 2.

1) Hirokawa (1).

Thus

$$G(t) = \sum_{v=1}^{\infty} (T_v - T_{v-1}) \sum_{n=v}^{\infty} c_{n-v}\, \rho_n(t)$$

$$= \sum_{v=1}^{\infty} (T_v - T_{v-1})\, B_v(t)$$

$$= \left( \sum_{v=1}^{n} + \sum_{v=n+1}^{\infty} \right) (T_v - T_{v-1})\, B_v(t)$$

$$= \Sigma_1' + \Sigma_2', \text{ say.} \tag{2.7.1}$$

We obtain the estimates for $\Sigma_1'$ and $\Sigma_2'$ in the same manner as that of $\Sigma_1$ and $\Sigma_2$ in the proof of Theorem 1.

From Lemma 8, we get

$$\left|\Sigma_2'\right| = \left| \sum_{v=n+1}^{\infty} (T_v - T_{v-1})\, B_v(t) \right|$$

$$= O\left( \sum_{v=n+1}^{\infty} \left| T_v - T_{v-1} \right| \frac{1}{vt\, P_\tau} \right)$$

$$= O\left(\frac{1}{vt\, P_\tau}\right) \left( \sum_{v=n+1}^{\infty} \frac{\sigma_v}{v(v+1)} - \frac{\sigma_n}{n+1} \right)$$

$$= O\left[\frac{\tau}{P_\tau} \cdot \frac{P_n}{n}\right] = O\left(\frac{P_\mu}{\mu}\right)$$

$$= O(1) \cdot \frac{P_\mu}{\mu}. \tag{2.7.2}$$

by hypothesis and Lemmas 1 and 2.

Next, we have

$$\Sigma_1' = \sum_{v=1}^{n} (T_v - T_{v-1})\, B_v(t)$$

$$= \sum_{v=1}^{n} (W_v - W_{v+1})\, v\, B_v(t)$$

$$= \sum_{v=1}^{n} W_v \left\{ vB_v(t) - (v-1)B_{v-1}(t) \right\} - nW_{n+1} B_n(t)$$

$$= -\sum_{v=1}^{n} W_v v \left\{ B_{v-1}(t) - B_v(t) \right\} + \sum_{v=1}^{n} W_v B_{v-1}(t) - nW_{n+1} B_n(t)$$

$$= -\Sigma'_{1,1} + \Sigma'_{1,2} - n\, W_{n+1}\, B_n(t),$$

where, by Lemmas 1, 3(ii) and 9,

$$\Sigma'_{1,1} = \sum_{v=1}^{n} v\, W_v\, \Delta B_{v-1}(t)$$

$$= \sum_{v=1}^{n} \left\{ \sum_{\mu=1}^{v} \mu W_\mu \right\} \Delta^2 B_{v-1}(t) + \Delta B_n(t) \sum_{v=1}^{n} v\, W_v$$

$$= O\left(nt \frac{P_n}{P_\tau}\right) + O\left(\frac{P_n}{P_\tau}\right)$$

$$= O(\mu P_\mu) + O(P_\mu)$$

$$= O(1).$$

Since $\sum_{v=1}^{m} v\, W_v = O\left(\sum_{v=1}^{n} v.\ \frac{P_v}{v}\right) = O(n\, P_n)$; and by applying Abel's transformation twice, writing $W'_m = \sum_{\mu=1}^{m} W_\mu$ and by virtue of Lemmas 1, 3(ii) and 9, we have

$$\Sigma'_{1,2} = \sum_{v=1}^{n}\left(\sum_{m=1}^{v} W'_m\right) \overset{2}{\Delta} B_{v-1}(t) + \Delta B_n(t) \sum_{v=1}^{n} W'_v + B_n(t) W'_n$$

$$= O\left(nt\, \frac{P_n}{P_\tau}\right) + O\left(\frac{P_n}{P_\tau}\right) + O\left(\frac{P_\mu}{\mu}\right)$$

$$= O(1)\ \mu P_\mu + O(1)\ P_\mu + O(1)\ \frac{P_\mu}{\mu}$$

$$= O(1)\ ;$$

and by Lemmas 1, 3(i) and 9, we have

$$n\, W_{n+1}\, B_n(t) = O(1)\ \frac{P_\mu}{\mu} = O(1).$$

Hence

$$\Sigma'_1 = O(1). \tag{2.7.3}$$

Therefore, from (2.7.1), (2.7.2) and (2.7.3), we obtain

$$\theta(t) = O(1)\ \frac{P_\mu}{\mu} + O(1), \text{ as } t \to 0.$$

Consequently, as in Theorem 1 .

$$\limsup_{t\to 0} |G(t)| \leq O(1).\ \frac{P_\mu}{\mu}.$$

$\mu$ being arbitrary large and $O(1)$ independent of $\mu$, we finally get

$$G(t) \to 0, \quad \text{as} \quad t \to 0.$$

This completes the proof of Theorem 2.

2.8. By an appeal to Lemma 5, we obtain the following interesting Corollaries from our theorems.

_Corollary I. If $\Sigma a_n$ is $(N, p_n)$- summable and if_

$$\sigma_n = \sum_{v=1}^{n} | T_v - T_{v-1} | = O(P_n),$$

_then $\Sigma a_n$ is $(R,1)$- summable, provided $\{p_n\}$ is non-negative, non-increasing sequence, such that, $p_0=1$, $P_n\to\infty$, and_

(i) $\left\{ \dfrac{p_{n+1}}{p_n} \right\}$ _is non-decreasing_ ;

(ii) $\displaystyle\sum_{v=n}^{\infty} \frac{P_{v-n}}{v(v+1)} = O\left(\frac{P_n}{n}\right)$ ;

(iii) $\displaystyle\sum_{v=0}^{n} \frac{1}{P_v} = O\left(\frac{n}{P_n}\right)$ ;

(iv) <u>for a positive integer</u> $\mu$ <u>and</u>

$$n = [\mu t^{-1}], \quad \tau = [t^{-1}],$$

$$P_n = O(P_\mu P_\tau).$$

<u>Corollary II.</u> <u>Under the hypotheses of Corollary I</u>, $\Sigma a_n$ <u>is summable</u> $(B_1)$.

# Chapter III

## TAUBERIAN THEOREMS FOR SUMMABILITY METHODS OF RIEMANN-TYPE(I)

3.1 Let $\Sigma a_n$ be a given infinite series with $s_n$ for its n-th partial sum, and let $S_n^{\alpha}$ denote the n-th Cesaro-sum of order $\alpha$ $(\alpha > -1)$ of the sequence $s_n$, defined by:

$$(3.1.1) \qquad S_n^{\alpha} = \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v = \sum_{v=0}^{n} A_{n-v}^{\alpha} a_v ,$$

where $A_n^{\alpha}$ is given by the relation:

$$(3.1.2) \qquad \sum_{n=0}^{\infty} A_n^{\alpha} x^n = (1-x)^{-\alpha-1}, \quad (|x| < 1).$$

The series $\Sigma a_n$ is said to be summable to sum $s$ by Riemann-Cesaro method of order 1 and index $\alpha$, $(-1 \leq \alpha \leq 0)$, or briefly summable $(R, 1, \alpha)$ to sum $s$, if the series

$$(3.1.3) \qquad F(\alpha, t) = C_{\alpha}^{-1} t^{\alpha+1} \sum_{n=1}^{\infty} S_n^{\alpha} \left(\frac{\sin nt}{nt}\right)$$

converges in some interval $0 < t < t_0$, and

$$\lim_{t \to 0} F(\alpha, t) = s,$$

where

$$C_\alpha = \begin{cases} \dfrac{1}{\Gamma(\alpha+1)} \displaystyle\int_0^\infty u^{\alpha-1} \sin u \, du & , \quad -1 < \alpha < 0 \; ; \\ \pi/2 & , \quad \alpha = 0 \; ; \\ 1 & , \quad \alpha = -1.^{1)} \end{cases}$$

Under this definition the summability method $(R, 1, \alpha)$ reduces to $(R, 1)$ and $(R_1)$-method for $\alpha = -1$ and $\alpha = 0$ respectively.

The method $(R, 1, \alpha)$ is not regular when $-1 \leq \alpha \leq 0$. [2)]

Other relevant definitions and notations are the same as given in Chapter I and II . In what follows we take $a_0 = 0$ , so that $r_0 = 0$.

3.2 Introduction. Since the method $(R, 1, \alpha)$ is not regular for $-1 \leq \alpha \leq 0$, i.e., the convergence of a series

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1) Hirokawa (1).

2) Hirokawa (1).

$\Sigma a_n$ to the sum $s$ does not imply its $(R, 1, \alpha)$ summability to the same sum. It is natural to ask the following question : Is it possible to prove the inclusion relation $(C, 0) \subset (R, 1, \alpha)$ under some Tauberian conditions ? The question was answered in the affirmative by Szász, Varshney and Singh , in the cases of $(R, 1)$ and $(R_1)$-summability methods. In Chapter II we have generalised the results of the aforesaid authors. In the present chapter we propose to extend the result of Chapter II for $(R, 1, \alpha)$-method of summation, and prove here a couple of analogous theorems for this method. Theorem 1 is a general one, while Theorem 2 gives a simplified result which can be easily applied.

3.3 We prove the following theorems :

<u>Theorem</u> 1 . <u>If</u> $\Sigma a_n$ <u>is</u> $(N, p_n)$-<u>summable and if</u> ,

$$(3.3.1) \qquad \sigma_n = \sum_{k=1}^{n} | \tau_k - \tau_{k-1} | = O(P_n) ;$$

<u>then the series</u> $\Sigma a_n$ <u>is summable by</u> $(R, 1, \alpha)$-<u>method for</u> $-1 \leq \alpha \leq 0$, <u>provided that</u> $\{p_n\}$ <u>is a non-negative</u> , <u>non-increasing</u>

<u>sequence such that</u> $P_n \to \infty$, <u>and</u>

$$\text{(i)}\quad d_n = \sum_{k=0}^{n} c_k = O\left(\frac{1}{P_n}\right);$$

$$\text{(ii)}\quad \sum_{k=n+1}^{\infty} |c_k| = O\left(\frac{1}{P_n}\right), \quad n \geq 0;$$

$$\text{(iii)}\quad \sum_{k=n}^{\infty} \frac{P_{k-n}}{k(k+1)} = O\left(\frac{P_n}{n}\right); \quad n \geq 1$$

$$\text{(iv)}\quad \sum_{k=0}^{n} \frac{1}{P_k} = O\left(\frac{n}{P_n}\right);$$

<u>and</u>

(v) <u>for a positive number</u> $\mu$ <u>and</u> $n = [\mu t^{-1}]$, $\tau = [t^{-1}]$,

$$P_n = O(P_\mu P_\tau).$$

Combining Theorem 1 with Lemma 8 below, we also get the following interesting and simple result.

<u>Theorem 2. Let</u> $\{p_n\}$ <u>be a positive, non-increasing sequence such that</u> $p_0 = 1$, $P_n \to \infty$, $\left\{\frac{p_{n+1}}{p_n}\right\}$ <u>is a non-decreasing sequence and the conditions</u> (iii) <u>through</u> (v) <u>hold. If</u> $\Sigma a_n$ <u>is</u> $(N, p_n)$-<u>summable, and if</u> (3.3.1) <u>holds, then</u> $\Sigma a_n$ <u>is also summable</u> $(R, 1, \alpha)$ <u>for</u> $-1 \leq \alpha \leq 0$.

3.4 We shall require the following lemmas for proving our theorems.

<u>Lemma 1.</u>[1] <u>If</u> $\{p_n\}$ <u>is a non-negative, non-increasing sequence such that the series</u>

$$\sum_{v=n}^{\infty} \frac{p_{v-n}}{v(v+1)}$$

<u>converges, then</u> $\frac{p_n}{n} \to 0$, <u>as</u> $n \to \infty$.

<u>Lemma 2.</u>[2] Let $\{p_n\}$ <u>be a non-negative, non-increasing sequence such that</u>

$$\sum_{v=n}^{\infty} \frac{p_{v-n}}{v(v+1)} = O\left(\frac{p_n}{n}\right) .$$

<u>Then, for</u> $n \geq 1$,

$$\sum_{v=n}^{\infty} \frac{p_v}{v(v+1)} = O\left(\frac{p_n}{n}\right) . \tag{3.4.1}$$

<u>Lemma 3.</u>[3] <u>Let</u> $\{p_n\}$ <u>be a non-negative, non-increasing sequence such that</u> $\left\{\frac{p_n}{n}\right\}$ <u>is a null sequence. If</u> $\Sigma\, a_n$ <u>is summable</u> $(N, p_n)$, <u>then</u>

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1) Chapter II, Lemma 1.

2) Chapter II, Lemma 2.

3) Chapter II, Lemma 3.

(i) $$w_n = \sum_{v=n}^{\infty} \frac{T_v - T_{v-1}}{v} = o\left(\frac{P_n}{n}\right),$$

(ii) $$w'_m = \sum_{v=1}^{m} w_v = o(P_m).$$

__Lemma 4.__[1] __Let__ $H_v(t) = t^{\alpha-1} \sum_{n=v}^{\infty} A_{n-v}^{\alpha-1} \frac{\sin nt}{nt}$. __Then__, __for__ $-1 \leq \alpha \leq 0$,

(3.4.2) $$H_v(t) = O(1/v),$$

(3.4.3) $$\eta_n(t) = \sum_{v=n}^{\infty} H_v(t) = O\left(\frac{1}{nt}\right),$$

and

(3.4.4) $$\Delta^m H_v(t) = O(v^{-1} t^m),$$

__where__ $m$ __is a non-negative integer and__ $\Delta^m H_v(t)$ __is the__ $m$__-th difference of__ $H_v(t)$ __with respect to__ $v$.

__Lemma 5.__[2] __If__ $s_n^1 = o(n)$, __then we have for__ $-1 \leq \alpha \leq 0$,

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1) Hirokawa (1), Lemmas 7, 8 and 9.

2) Hirokawa (2), Lemma 7.

$$t^{\alpha+1} \sum_{n=1}^{\infty} S_n^{\alpha} \frac{\sin nt}{nt} = \sum_{n=1}^{\infty} a_n H_n(t).$$

<u>Lemma</u> 6. <u>If</u> $a_n \eta_{n+1}(t) = O(1)$ , $n \to \infty$ <u>and</u> $\sum_{n=1}^{\infty} a_n \eta_n(t) < \infty$

<u>then</u>

(3.4.5) $$\sum_{n=1}^{\infty} a_n H_n(t) = \sum_{n=1}^{\infty} a_n \eta_n(t)$$

The lemma follows from the identity :

$$\sum_{\nu=1}^{m} a_\nu H_\nu(t) = \sum_{1}^{m} a_\nu \eta_\nu(t) - a_m \eta_{m+1}(t).$$

<u>Lemma</u> 7.[1] <u>If</u> $p(x) = \sum p_n x^n$ <u>is convergent for</u> $|x| < 1$,

<u>and</u>

(3.4.6) $$p_0 = 1,\ p_n > 0,\ \frac{p_{n+1}}{p_n} \geq \frac{p_n}{p_{n-1}} \quad ( n > 0 ),$$

<u>then</u>

(3.4.7) $$\{p(x)\}^{-1} = 1 + c_1 x + c_2 x^2 + \dots$$

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1) Hardy (1) , Theorem 22.

*where* $c_n \leq 0$ *, for* $n = 1, 2, \ldots$ *,* $\sum_{n=1}^{\infty} |c_n| \leq 1$ *. If* $\sum p_n = \infty$*, then* $\sum_{n=1}^{\infty} |c_n| = 1$.

*Lemma 8.*[1] *If* $\{p_n\}$ *is a positive and non-increasing sequence such that* $p_0 = 1$ *,* $P_n \to \infty$ *, and* $\{p_{n+1}/p_n\}$ *, is non-decreasing sequence, then*, for $n > 0$ ,

$$d_n = \sum_{v=n+1}^{\infty} |c_v| = \sum_{v=0}^{n} c_v = O\left(\frac{1}{P_n}\right).$$

*Remark.* The identity

$$d_n = \sum_{v=n+1}^{\infty} |c_v| = \sum_{v=0}^{n} c_v$$

is obtained by virtue of the Lemma 7.

*Lemma 9 . If* $\{p_n\}$ *is such that it satisfies all the conditions of the theorem except* (iii), *then the series*

(3.4.6) $$\sum_{n=0}^{\infty} c_n \, \eta_{n+v}(t) = E_v(t),$$

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1) Varshney (1) , Lemma 2.

<u>is absolutely convergent and for</u> $m = 0, 1, 2, \ldots$ , <u>we have</u>

$$(3.4.9) \qquad \Delta^{m} E_{v}(t) = O\left(\frac{t^{m-1}}{v\, P_{\tau}}\right) .$$

<u>Proof</u>. Absolute convergence of the series (3.4.8) follows from the hypothesis (ii) , since $\sum_{n=1}^{\infty} |c_n| < \infty$. To prove (3.4.9) , for $m = 1, 2, \ldots$ , we have

$$\begin{aligned}
\Delta^{m} E_{v}(t) &= \Delta_{v}^{m}\Big( \sum_{n=0}^{\infty} c_n\, \eta_{n+v}(t)\Big) \\
&= \Delta_{v}^{m-1}\Big( \sum_{n=0}^{\infty} c_n\, \Delta_{v}\, \eta_{n+v}(t) \Big) \\
&= \Delta_{v}^{m-1}\Big( \sum_{n=0}^{\infty} c_n\, H_{n+v}(t) \Big) \\
&= \sum_{n=0}^{\infty} c_n\, \Delta_{v}^{m-1}\big( H_{n+v}(t) \big) \\
&= \Big( \sum_{n=0}^{\tau} + \sum_{n=\tau+1}^{\infty} \Big)\, c_n\, \Delta_{v}^{m-1} H_{n+v}(t) \\
&= E_{v}^{(1)}(t) + E_{v}^{(2)}(t) \text{ , say .}
\end{aligned}$$

Now, by hypothesis (ii) and Lemma 4, we have for $m = 1, 2, \ldots$

$$E_v^{(2)}(t) = \sum_{n=\tau+1}^{\infty} c_n \, \Delta_v^{m-1} H_{n+v}(t)$$

$$= O\left( \sum_{n=\tau+1}^{\infty} |c_n| \, \frac{t^{m-1}}{(n+v)} \right)$$

$$= O\left( \frac{t^{m-1}}{v+\tau+1} \sum_{n=\tau+1}^{\infty} |c_n| \right)$$

$$= O\left( \frac{t^{m-1}}{v \, P_\tau} \right).$$

By Abel's transformation and Lemma 4 and hypothesis (i), we have, for $m = 1, 2, \ldots$ ,

$$E_v^{(1)}(t) = \sum_{n=0}^{\tau-1} d_n \Delta_n \left( \Delta_v^{m-1} H_{n+v}(t) \right) + d_\tau \Delta_v^{m-1} H_{\tau+v}(t) ,$$

$$= \sum_{n=0}^{\tau-1} d_n \, \Delta^{m} H_{n+v}(t) + O\left( \frac{t^{m-1}}{v \, P_\tau} \right)$$

$$= O\left( \sum_{n=0}^{\tau-1} \frac{1}{P_n} \cdot \frac{t^{m}}{(n+v)} \right) + O\left( \frac{t^{m-1}}{v \, P} \right)$$

$$= O\left( \frac{t^{m-1}}{v \, P_\tau} \right) ,$$

by hypothesis (iv).

By Lemma 4, for $m = 0$ , we have

$$\Delta^{m} E_v(t) = E_v(t) = \sum_{n=0}^{\infty} c_n \, \eta_{n+v}(t)$$

$$= \sum_{n=0}^{\tau} c_n \eta_{n+v}(t) + \sum_{n=\tau+1}^{\infty} c_n \eta_{n+v}(t)$$

$$= \sum_{n=0}^{\tau} d_n H_{n+v}(t) + d_{\tau}\eta_{n+v+1}(t) + O(1)\left(\frac{1}{vt}\sum_{n=\tau+1}^{\infty} |c_n|\right)$$

$$= O\left(\frac{1}{v}\sum_{n=0}^{\tau}\frac{1}{P_n}\right) + O\left(\frac{1}{vt\,P_{\tau}}\right) + O\left(\frac{1}{vt\,P_{\tau}}\right)$$

$$= O\left(\frac{1}{vt\,P_{\tau}}\right),$$

by hypotheses. Hence the result.

<u>Lemma 10. Let</u> $\{p_n\}$ <u>be a non-negative sequence such that</u> $P_n \to \infty$, <u>and the conditions</u> (i) <u>and</u> (iv) <u>of the Theorem 1 hold. Then</u> $(N, p_n)$-<u>summability of the series</u> $\Sigma a_n$ <u>to the sum</u> $s$ <u>implies its</u> $(C, 1)$-<u>summability to the same sum. In particular, if</u> $T_n = o(P_n)$, <u>then</u> $S_n^1 = o(n)$.

<u>Proof</u>. Let us denote by $s_n^1$ the n-th Cesàro-mean of order 1 of the sequence $\{s_n\}$. Then by definition, and using (2.2.4)

$$s_n^1 - s = \frac{1}{n}\sum_{v=0}^{n}(s_v - s)$$

$$= \frac{1}{n}\sum_{v=0}^{n}\sum_{\mu=0}^{v} c_{v-\mu}\left(\frac{t_\mu}{P_\mu} - s\right)P_\mu$$

$$= \frac{1}{n} \sum_{\mu=0}^{n} (t_\mu - s) P_\mu \, d_{n-\mu}$$

$$= o(1) \frac{1}{n} \sum_{\mu=0}^{n} P_\mu \left| d_{n-\mu} \right|$$

$$= o(1) \cdot O\left( \frac{P_n}{n} \right) \sum_{\mu=0}^{n} \frac{1}{P_\mu}$$

$$= o(1) \; O\left( \frac{P_n}{n} \cdot \frac{n}{P_n} \right) = o(1)$$

by hypotheses.

Now, if we take $s = 0$, then we get the second part of the lemma.

This terminates the proof of Lemma 10.

3.5 <u>Proof of Theorem 1</u>. As in Chapter II, we may assume without loss of generality that $T_n = o(P_n)$, as $n \to \infty$. By Lemmas 10 and 5, we have

$$t^{\alpha+1} \sum_{n=1}^{\infty} S_n^\alpha \frac{\sin nt}{nt} = \sum_{n=1}^{\infty} s_n \, H_n(t).$$

Now, since from (2.2.4), for fixed $t$,

$$s_n \, \eta_{n+1}(t) = \eta_{n+1}(t) \sum_{v=1}^{n} c_{n-v} \, T_v$$

$$= O\left( \frac{P_n}{nt} \sum_{v=1}^{n} |c_{n-v}| \right) = O(1) ,$$

by hypotheses (ii) and (iii) and Lemma 1. Therefore by virtue of Lemma 6, it will be enough to show that $\sum_{n=1}^{\infty} a_n \eta_n(t)$ converges in $0 < t < t_0$.

Using (2.2.5), we observe that

$$\sum_{n=1}^{\infty} a_n \eta_n(t) = \sum_{n=1}^{\infty} \left( \sum_{v=1}^{n} c_{n-v} ( T_v - T_{v-1} ) \eta_n(t) \right)$$

$$= \sum_{v=1}^{\infty} ( T_v - T_{v-1} ) \sum_{n=v}^{\infty} c_{n-v} \eta_n(t) ,$$

the interchange of order of summation is justified, as proved below the double series is absolutely convergent.

Since, by hypothesis (ii), $\sum_{n=0}^{\infty} |c_n| < \infty$, and the fact that $\eta_n(t) = O(\frac{1}{nt})$ (by Lemma 4) for every fixed $t > 0$ , we see that

$$\sum_{v=1}^{\infty} | T_v - T_{v-1} | \sum_{n=0}^{\infty} | c_{n-v} \eta_{n+v}(t) |$$

$$= O\left( \sum_{v=1}^{\infty} \frac{1}{v} | T_v - T_{v-1} | \right).$$

Now, as $n \to \infty$ ,

$$\sum_{v=1}^{m} 1/v \left| T_v - T_{v-1} \right| = O(m^{-1}\sigma_m) + O\left( \sum_{v=1}^{m-1} \frac{\sigma_v}{v(v+1)} \right)$$

$$= O(1)$$

by hypotheses and Lemmas 1 and 2.

Thus

$$F(\alpha,t) = \sum_{v=1}^{\infty} ( T_v - T_{v-1}) \sum_{n=v}^{\infty} c_{n-v}\, \eta_n(t)$$

$$= \left( \sum_{v=1}^{n} + \sum_{v=n+1}^{\infty} \right) ( T_v - T_{v-1}) E_v(t)$$

$$(3.5.1) \qquad = \psi_1 + \psi_2 \text{, say.}$$

Using Lemmas 1, 2 and 9 and proceeding as in the proof of Theorem 1 of Chapter II, it can be easily shown that

$$|\psi_2| = \left| \sum_{v=n+1}^{\infty} (T_v - T_{v-1}) E_v(t) \right|$$

$$(3.5.2) \qquad = O(1) \frac{p_\mu}{\mu} \text{.}$$

by hypothesis (v) .

Next, the proof of the estimate $\psi_1$ can be constructed on the basis of that of $\Sigma_1$ of Theorem 1 in Chapter II. However, for the sake of completeness we give it here.

$$\psi_1 = \sum_{v=1}^{n} (T_v - T_{v-1}) E_v(t) = \sum_{v=1}^{n} (W_v - W_{v+1}) \, v \, E_v(t)$$

$$= -\sum_{v=1}^{n} W_v \, v \, \Delta E_{v-1}(t) + \sum_{v=1}^{n} W_v \, E_{v-1}(t) - nW_{n+1} E_n(t).$$

$$= -\psi_{1,1} + \psi_{1,2} - n\, W_{n+1}\, E_n(t),$$

where by Lemmas 1, 2, 3(ii) and 9

$$\psi_{1,1} = \sum_{v=1}^{n} \left\{ \sum_{\mu=1}^{v} \mu W_\mu \right\} \left\{ \Delta^2 E_{v-1}(t) \right\} + \Delta E_n(t) \sum_{v=1}^{n} v\, W_v$$

$$= O\left( \sum_{v=1}^{n} v\, P_v \frac{1}{vP_\tau} \right) + O\left( \frac{1}{nP_\tau} \cdot n\, P_n \right)$$

$$= O(1),$$

since $\sum_{v=1}^{n} v\, W_v = O(n\, P_n)$ ; and by applying Abel's transformation twice, and by virtue of Lemmas 3(ii) and 9, we have

$$\psi_{1,2} = \sum_{v=1}^{n} \left( \sum_{k=1}^{v} W'_k \right) \Delta^2 E_{v-1}(t) + \Delta E_n(t) \sum_{v=1}^{n} W'_v + E_n(t) W'_n$$

$$= O\left( nt \frac{P_n}{P_\tau} \right) + O\left( \frac{P_n}{P_\tau} \right) + O\left( \frac{P_\mu}{\mu} \right)$$

$$= O(1),$$

by hypothesis; and by Lemmas 1, 3(i) and 9, we get

$$n\, W_{n+1}\, E_n(t) = o(1).$$

Hence,

$$\psi_1 = o(1) . \tag{3.5.3}$$

Therefore, from (3.5.1), (3.5.2) and (3.5.3) it follows that

$$F(a,t) = O(1)\,\frac{F_\mu}{\mu} + o(1) , \quad \text{as } t \to 0.$$

Hence,

$$\limsup_{t\to 0} \left| F(a,t) \right| \leq O(1)\,\frac{F_\mu}{\mu} ,$$

being arbitrary large and $O(1)$ independent of $\mu$, we ultimately get

$$F(a, t) \to 0, \quad \text{as } t \to 0 .$$

This terminates the proof of Theorem 1.

## Chapter IV

## TAUBERIAN THEOREMS FOR SUMMABILITY METHODS OF RIEMANN TYPE (II).

4.1 Let $\Sigma a_n$ be a given infinite series with the sequence of partial sums $s_n$, where $s_n = a_0 + a_1 + \dots + a_n$; and let $S_n^\alpha$ denote the n-th Cesaro-sum of order $\alpha$ $(\alpha > -1)$ of the sequence $\{s_n\}$, defined by :

$$(4.1.1) \quad S_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha} a_\nu ,$$

where $A_n^\alpha$ is given by the relation :

$$(4.1.2) \quad \sum_{n=0}^{\infty} A_n^\alpha x^n = (1-x)^{-\alpha-1} , \qquad (|x| < 1 ).$$

The series $\Sigma a_n$ is said to be summable $(K, 1, \alpha)$ to the sum $s$, if the series

$$f(\alpha,t) = B_\alpha^{-1} t^{\alpha+1} \sum_{n=1}^{\infty} S_n^\alpha \int_t^\pi \frac{\sin nx}{2\tan x/2}\, dx$$

converges in some interval $0 < t < t_0$, and if

$$\lim_{t \to 0+} f(\alpha, t) = s,$$

where

$$B_\alpha = \begin{cases} \pi/2 & , \quad \alpha = -1 ; \\ (\alpha+1)^{-1} \sin (\alpha+1)\pi/2 & , \quad -1 < \alpha < 0 ; \\ 1 & , \quad \alpha = 0. \end{cases}$$ [1)]

When $\alpha = -1$, the method $(K, 1, \alpha)$ reduces to the method $(k, 1)$.[2)]

The method $(K, 1, \alpha)$ is not regular when $-1 \leq \alpha \leq 0$.[3)]

Other relevant definitions and notations are the same as given in Chapters I and II. In what follows we take $a_0 = 0$, so that $T_0 = 0$.

4.2 Introduction. It has been proved by Izumi[4)] that, for Fourier series, summability $(K,1)$ is equivalent to summability $(R_1)$. Since it is known that for Fourier series summability methods

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1) Hirokawa (3).

2) Zygmund (1).

3) Hirokawa (3).

4) Izumi (1).

(N, 1) and $(R_1)$ are mutually exclusive [1] it follows that, in general summability methods (K, 1) and (R, 1) are also independent of each other. We also know [2] that $(K, 1, \alpha)$-method of summation have some properties similar to that of $(R, 1, \alpha)$. Therefore, the object of this chapter is to establish two theorems for $(K, 1, \alpha)$-summability method analogous to that proved for $(R, 1, \alpha)$ summability in the preceding chapter (Chapter III).

4.3 We prove the following theorems :

*Theorem 1. If* $\Sigma a_n$ *is* $(N, p_n)$*-summable and if*

$$(4.3.1)\qquad \sum_{k=1}^{n} |T_k - T_{k-1}| = O(P_n),$$

*then the series* $\Sigma a_n$ *is summable by* $(K, 1, \alpha)$*-method for* $-1 \le \alpha \le 0$, *provided that* $\{p_n\}$ *is a non-negative, non-increasing sequence, such that* $P_n \to \infty$, *and*

$$(4.3.2)\qquad d_n = \sum_{k=0}^{n} c_k = O\left(\frac{1}{P_n}\right) ;$$

$$(4.3.3)\qquad \sum_{k=n+1}^{\infty} |c_k| = O\left(\frac{1}{P_n}\right), \quad n \ge 0 ;$$

1) Hardy and Rogosinski (1).

2) Hirokawa (3).

$$(4.3.4) \qquad \sum_{k=n}^{\infty} \frac{P_{k-n}}{k(k+1)} = O\left(\frac{P_n}{n}\right), \quad n \geq 1 ;$$

$$(4.3.5) \qquad \sum_{k=0}^{n} \frac{1}{P_k} = O\left(\frac{n}{P_n}\right) ;$$

and

(4.3.6) *for a positive integer* $\mu$ *and* $n = [\mu t^{-1}]$, $\tau = [t^{-1}]$,

$$P_n = O(P_\mu \, P_\tau).$$

Combining Theorem 1 with Lemma 5 below, we also get the following interesting and simple result, corresponding to that of Theorem 2 of Chapter III .

*Theorem 2. Let* $\{p_n\}$ *be a positive, non-increasing sequence, such that* $p_0 = 1$, $P_n \to \infty$, $\left\{\frac{p_{n+1}}{p_n}\right\}$ *is a non-decreasing sequence and the conditions* (4.3.4) *through* (4.3.6) *hold. If* $\Sigma a_n$ *is* $(N, p_n)$*-summable and, if* (4.3.1) *holds, then* $\Sigma a_n$ *is also summable* $(N, 1, \alpha)$ *for* $-1 \leq \alpha \leq 0$.

4.4 We need the following lemmas for the proof of our theorems.

*Lemma 1.*[1] *If* $\{p_n\}$ *is a non-negative, non-increasing sequence such that the series*

$$\sum_{v=n}^{\infty} \frac{p_{v-n}}{v(v+1)}$$

*converges, then* $\frac{p_n}{n} \to 0$, *as* $n \to \infty$.

*Lemma 2.*[2] *Let* $\{p_n\}$ *be a non-negative, non-increasing sequence such that, for* $n \geq 1$,

$$\sum_{v=n}^{\infty} \frac{p_{v-n}}{v(v+1)} = O\left(\frac{p_n}{n}\right).$$

*Then, for* $n \geq 1$,

$$\sum_{v=n}^{\infty} \frac{p_v}{v(v+1)} = O\left(\frac{p_n}{n}\right). \tag{4.4.1}$$

*Lemma 3.*[3] *Let* $\{p_n\}$ *be a non-negative, non-increasing sequence such that* $\left\{\frac{p_n}{n}\right\}$ *is a null suquence. If* $\Sigma a_n$ *is summable* $(N, p_n)$, *then*

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1) Chapter II, Lemma 1.

2) Chapter II, Lemma 2.

3) Chapter II, Lemma 3.

$$(i) \quad W_n = \sum_{v=n}^{\infty} \frac{T_v - T_{v-1}}{v} = O\left(\frac{P_n}{n}\right),$$

$$(ii) \quad W'_n = \sum_{v=1}^{n} W_v = O(P_n).$$

<u>Lemma 4.</u>[1] <u>If</u> $p(x) = \sum p_n x^n$ <u>is convergent for</u> $|x| < 1$, <u>and</u>

$$(4.4.2) \qquad p_0 = 1,\ p_n > 0,\ \frac{p_{n+1}}{p_n} \geq \frac{p_n}{p_{n-1}} \qquad (n > 0),$$

<u>then</u>

$$(4.4.3) \qquad \{p(x)\}^{-1} = 1 + c_1 x + c_2 x^2 + \ldots$$

<u>where</u> $c_n \leq 0$, <u>for</u> $n = 1, 2, \ldots$, $\sum_{n=1}^{\infty} |c_n| \leq 1$. <u>If</u> $\sum p_n = \infty$, <u>then</u> $\sum_{n=1}^{\infty} |c_n| = 1$.

<u>Lemma 5.</u>[2] <u>If</u> $\{p_n\}$ <u>is a positive and non-increasing sequence such that</u> $p_0 = 1$, $P_n \to \infty$, <u>and</u> $\{p_{n+1}/p_n\}$ <u>is a non-decreasing sequence, then for</u> $n > 0$,

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1) Hardy (1), Theorem 22.

2) Varshney (1), Lemma 2.

$$d_n = \sum_{v=n+1}^{\infty} |c_n| = \sum_{v=0}^{n} c_v = O\left(\frac{1}{P_n}\right).$$

Remark. The identity

$$d_n = \sum_{v=n+1}^{\infty} |c_v| = \sum_{v=0}^{n} c_v$$

is obtained by virtue of the Lemma 4.

Lemma 6.[1] Let $p_n$ be a non-negative sequence such $P_n \to \infty$, and the conditions (4.3.2) and (4.3.3) of the Theorem 1 hold. Then $(N, p_n)$-summability of the series $\Sigma a_n$ to the sum $s$ implies its $(C, 1)$-summability to the same sum. In particular if $T_n = O(P_n)$, then $S_n^1 = o(n)$.

Lemma 7. Let $\phi(n,t) = \int_t^{\pi} \frac{\sin nu}{2\tan {}^u/_2}\, du$. Then

(4.4.4) $$\phi(n, t) = O\left(\frac{1}{nt}\right)$$

and

(4.4.5) $$\Delta^m \phi(n,t) = O\left(\frac{t^{-m-1}}{n}\right),$$

where $\Delta^m \phi(n,t)$ denotes the $m$-th difference of $\phi(n,t)$ with respect to $n$, and $m$ is a non-negative integer.

Proof. $$\phi(n,t) = \int_t^{\pi} \frac{\sin nu}{2\tan {}^u/_2}\, du$$

1) Chapter III, Lemma 10.

$$= (2\tan {}^t/_2)^{-1} \int_t^{\xi} \sin nu\, du, \; t < \xi < \pi$$

$$= (2\tan {}^t/_2)^{-1} \left[ -\frac{\cos nu}{n} \right]_t^{\xi}$$

$$= O\left(\frac{1}{nt}\right).$$

Again

$$\Delta \phi(n,t) = \Delta \left[ \int_t^{\pi} \frac{\sin nu}{2\tan {}^u/_2}\, du \right]$$

$$= \int_t^{\pi} \frac{\sin nu - \sin(n+1)u}{2\tan {}^u/_2}\, du$$

$$= -\int_t^{\pi} \frac{2\cos(n+{}^1/_2)u \, \sin {}^u/_2}{2\tan {}^u/_2}\, du$$

$$= -\frac{1}{2} \int_t^{\pi} (\cos\,(n+1)u + \cos nu\,)\, du$$

$$= {}^t/_2 \left[ \frac{\sin(n+1)t}{(n+1)t} + \frac{\sin nt}{nt} \right].$$

Hence

$$\overset{m}{\Delta}\, \phi(n,t) = \overset{m-1}{\Delta}\; \Delta\phi(n,t) = \frac{t}{2}\; \overset{m-1}{\Delta} \left[ \frac{\sin(n+1)t}{(n+1)t} + \frac{\sin nt}{nt} \right]$$

$$= O\left(\frac{t^{m-1}}{n}\right)$$

by virtue of the fact that

$$\Delta^{m}\left(\frac{\sin nt}{nt}\right)^{p} = O\left(n^{-p}\, t^{m-p}\right).$$[1]

This completes the proof.

<u>Lemma 8</u>. <u>Let</u> $G_\nu(t) = t^{\alpha+1} \sum_{n=\nu}^{\infty} A_{n-\nu}^{\alpha-1}\, \phi(n,t)$, $-1 \leq \alpha \leq 0$.

Then

(4.4.6) $$G_\nu(t) = O(1/\nu),$$

<u>and for positive integer</u> $k$,

(4.4.7) $$\Delta^{k} G_\nu(t) = O\left(\frac{t^{k}}{\nu}\right).$$

The proof is analogous to that of a lemma of Hirokawa.[2]

Let $G_\nu(t) = t^{\alpha+1}\left(\sum_{n=\nu}^{\nu+\rho} + \sum_{n=\nu+\rho+1}^{\infty}\right) = U_1 + U_2$, say,

where $\rho = [1/t]$.

Now by (4.4.5) we have for $-1 < \alpha < 0$,

$$U_2 = t^{\alpha+1} \sum_{n=\nu+\rho+1}^{\infty} A_{n-\nu}^{\alpha-1}\, \phi(n,t)$$

$$= O\left(t^{\alpha}\, (\nu+\rho+1)^{-1} \sum_{n=\nu+\rho+1}^{\infty} (n-\nu)^{\alpha-1}\right)$$

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1) Obreshkoff (1), Lemma 3.

2) Hirokawa (1), Lemma 4.

$$= O(t^{\alpha} v^{-1} \rho^{\alpha}) = O(v^{-1}),$$

and on applying Abel's transformation to $U_1$ we have

$$U_1 = t^{\alpha+1} \sum_{n=0}^{\rho} A_n^{\alpha-1} \phi(n+v,t)$$

$$= t^{\alpha+1} \sum_{n=0}^{\rho-1} A_n^{\alpha} \Delta\phi(n+v,t) + t^{\alpha+1} A_\rho^{\alpha} \phi(\rho+v,t)$$

$$= O\left( t^{\alpha+1} \sum_{n=0}^{\rho-1} A_n^{\alpha} (n+v)^{-1} \right) + O(v^{-1})$$

$$= O\left( t^{\alpha+1} \rho^{\alpha+1} v^{-1} \right) + O(v^{-1})$$

$$= O(v^{-1}).$$

Hence $G_v(t) = O(v^{-1})$, for $-1 < \alpha < 0$. When $\alpha = 0$ $G_v(t) = t\,\phi(v,t) = O(v^{-1})$. Similarly, we have the result when $\alpha = -1$.

Now

$$G_v(t) = t^{\alpha+1} \sum_{n=v}^{\infty} A_{n-v}^{\alpha-1} \phi(n,t)$$

$$= t^{\alpha+1} \sum_{n=0}^{\infty} A_n^{\alpha-1} \phi(n+v,t).$$

Hence by using the method of proof of (4.4.6), we have

$$\overset{k}{\Delta}\, G_v(t) = t^{\alpha-1} \sum_{n=0}^{\infty} A_n^{\alpha-1} \overset{k}{\Delta}\, \phi(n+v,\ t)$$

$$= O\left(\frac{t^k}{v}\right),$$

Hence the lemma.

<u>Lemma</u> 9. <u>Let</u> $K_v(t) = \sum_{n=v}^{\infty} G_n(t)$. <u>Then</u>

(4.4.8) $$K_v(t) = O(v^{-1}\ t^{-1}).$$

<u>Proof</u>. We have

$$K_v(t) = \sum_{n=v}^{\infty} G_n(t) = t^{\alpha+1} \sum_{n=v}^{\infty} \sum_{k=n}^{\infty} A_{k-n}^{\alpha-1}\, \phi(k,t)$$

$$= t^{\alpha+1} \sum_{n=v}^{\infty} \sum_{k=0}^{\infty} A_k^{\alpha-1}\, \phi(n+k,\ t),$$

$$= t^{\alpha+1} \sum_{k=0}^{\infty} A_k^{\alpha-1} \sum_{n=v}^{\infty} \phi(n+k,\ t),$$

the change of order of summations can be easily justified. To prove the lemma, we just show that

$$Y_{v+k}(t) = \sum_{n=v}^{\infty} \phi(n+k,\ t) = O((v+k)^{-1}\ t^{-2}).$$

We have

$$\sum_{n=v}^{\infty} \phi(n+k,t) = \sum_{n=v}^{\infty} \int_t^{\pi} \frac{\sin((n+k)x)}{2\tan x/2}$$

$$= \sum_{n=v}^{\infty} \frac{1}{2\tan t/2} \int_t^{\xi} \sin((n+k)x)\,dx\ , \quad t<\xi<\pi$$

$$= (2\tan t/2)^{-1} \sum_{n=v}^{\infty} \left[ - \frac{\cos((n+k)x)}{(n+k)} \right]_t^{\xi}$$

$$= O\left(\frac{t^{-2}}{v+k}\right),$$

since $\sum_{n=v}^{\infty} \frac{\cos nt}{n} = O\left(\frac{1}{nt}\right)$.

Now, for $-1 < \alpha < 0$, we write

$$t^{\alpha+1} \sum_{k=0}^{\infty} A_k^{\alpha-1} \Upsilon_{v+k}(t) = \sum_{k=0}^{\rho} + \sum_{k=\rho+1}^{\infty} = V_1 + V_2\ , \text{ say.}$$

We have

$$V_2 = O\left( t^{\alpha+1} \sum_{k=\rho+1}^{\infty} k^{\alpha-1} (v+k)^{-1} t^{-2} \right)$$

$$= O\left( (v+\rho+1)^{-1} t^{\alpha-1} \rho^{\alpha} \right)$$

$$= O\left(v^{-1} t^{-1} \right);$$

and

$$V_1 = t^{\alpha+1} \sum_{k=0}^{\rho-1} A_k^{\alpha} \Delta_k (\Upsilon_{k+v}(t) + t^{\alpha-1} A_\rho^{\alpha} \Upsilon_{\rho+v}(t) )$$

$$= O(v^{-1}\, t^{-1}).$$

Hence (4.4.8) follows for $-1 < \alpha < 0$. The result for $\alpha = 0, -1$ is quite obvious. This completes the proof.

__Lemma__ 10. __If__ $S_n^1 = O(n)$, __then we have__

$$t^{\alpha+1} \sum_{n=1}^{\infty} S_n^{\alpha}\, \phi(n,t) = \sum_{n=1}^{\infty} a_n\, \sigma_n(t),$$

__where__

$$\sigma_n(t) = t^{\alpha+1} \sum_{v=n}^{\infty} A_{v-n}^{\alpha-1}\, \phi(v,t)\ , \quad (-1 \leq \alpha \leq 0).$$

__Proof.__ We have

$$t^{\alpha+1} \sum_{n=1}^{\infty} S_n^{\alpha}\, \phi(n,t) = t^{\alpha+1} \sum_{n=1}^{\infty} \phi(n,t) \sum_{k=1}^{n} A_{n-k}^{\alpha-1}\, a_k\ ,$$

$$= t^{\alpha+1} \sum_{k=1}^{\infty} a_k \sum_{n=k}^{\infty} A_{n-k}^{\alpha-1}\, \phi(n,t)$$

$$= \sum_{k=1}^{\infty} a_k\, \sigma_k(t)\ .$$

Here we shall prove that the change of order of summations is justified. For this purpose it is sufficient to prove that, for fixed $t > 0$,

$$I_n = \sum_{k=1}^{N} a_k \sum_{n=N+1}^{\infty} A_{n-k}^{\alpha-1} \phi(n,t) = O(1)\ , \text{ as } N \to \infty .$$

Using Abel's transformation, we have

$$I_n = \sum_{k=1}^{N-1} S_k^{1} \sum_{n=N+1}^{\infty} A_{n-k}^{\alpha-2} \phi(n,t) + S_N^{1} \sum_{n=N+1}^{\infty} A_{n-N}^{\alpha-1} \phi(n,t)$$

$$= \left( \sum_{k=1}^{N-1} |S_k^1|\, N^{-1} \cdot (N-k)^{\alpha-1} \right) + O\left(N N^{-1}\right)$$

$$= O(1), \text{ as } N \to \infty .$$

This proves the lemma.

<u>Lemma</u> 11. <u>Let</u> $G_n(t)$ <u>and</u> $K_n(t)$ <u>be the same as in Lemmas</u> 8 <u>and</u> 9 <u>respectively</u>. <u>If</u> $a_n K_{n+1}(t) = O(1)$, $n \to \infty$, <u>then the convergence of</u> $\sum_{n=1}^{\infty} a_n K_n(t)$ <u>implies the convergence of</u> $\sum_{n=1}^{\infty} a_n G_n(t)$ <u>and</u>

$$\sum_{n=1}^{\infty} a_n K_n(t) = \sum_{n=1}^{\infty} a_n G_n(t).$$

The proof of this lemma follows from the identity

$$\sum_{\nu=1}^{m} a_\nu G_\nu(t) = \sum_{\nu=1}^{m} a_\nu K_\nu(t) - a_m K_{m+1}(t).$$

<u>Lemma 12. If</u> $p_n$ <u>is such that it satisfies all the conditions of Theorem</u> 1, <u>except</u> (4.3.4) , <u>then the series</u>

$$(4.4.9) \qquad \sum_{n=0}^{\infty} c_n K_{n+v}(t) = Z_v(t) ,$$

<u>is absolutely convergent and , for</u> $m = 0, 1, 2, \ldots$ ,

$$(4.4.10) \qquad \overset{m}{\Delta} Z_v (t) = O\left( \frac{t^{m-1}}{v P_\tau} \right) ,$$

<u>where</u> $\overset{m}{\Delta} Z_v(t)$ <u>denote the</u> m-th <u>difference of</u> $Z_v(t)$ , <u>with respect to</u> $v$.

Proceeding in a manner parallel to that of the proof of Lemma 9 of Chapter III , we can easily prove this lemma .

4.5 <u>Proof of Theorem 1</u>. For the proof we may assume , without loss of generality , that $T_n = o\,(P_n)$ , as $n \to \infty$ . The proof of this theorem runs parallel to that of the proof of Theorem 1, Chapter III . Therefore, we give here only an outline of the proof, and omit the proof in detail. We have by virtue of Lemmas 6 and 10 ,

$$t^{\alpha+1} \sum_{n=1}^{\infty} S_n^{\alpha} \phi(n,t) = \sum_{n=1}^{\infty} a_n G_n(t).$$

Again by (2.2.4) and Lemma 9, we have as $n \to \infty$

$$a_n K_{n+1}(t) = K_{n+1}(t) \sum_{v=1}^{n} c_{n-v} \tau_v$$

$$= O\left[ \left(\frac{P_n}{(n+1)t}\right) \sum_{v=1}^{n-1} c_{n-v} \right] + O\left(\frac{P_n}{nt}\right)$$

$$= o(1) ,$$

for fixed $t > 0$, and by ( 4.3.3 ) , ( 4.3.4 ) and Lemma 1.

Therefore, by virtue of Lemma 11, we need only to show that $\Sigma\, a_n K_n(t)$ converges in the interval $0 < t < t_0$.

Employing (2.2.5), we have after a change of order of summations

$$\sum_{n=1}^{\infty} a_n K_n(t) = \sum_{n=1}^{\infty} K_n(t) \sum_{v=1}^{\infty} c_{n-v} (\tau_v - \tau_{v-1})$$

$$= \sum_{v=1}^{\infty} (\tau_v - \tau_{v-1}) \sum_{n=v}^{\infty} c_{n-v} K_n(t),$$

It can be easily shown that the interchange of order of summations being legitimate , since the double series is absolutely convergent.

Since by hypothesis and the fact that $\sum_{n=0}^{\infty} |c_n| < \infty$ for every fixed $t > 0$, we have

$$\sum_{v=1}^{\infty} |T_v - T_{v-1}| \sum_{n=0}^{\infty} |c_n K_{n+v}(t)|$$

$$= O\left( \sum_{v=1}^{\infty} 1/v \, |T_v - T_{v-1}| \right).$$

Now, as $m \to \infty$

$$\sum_{v=1}^{m} 1/v \, |T_v - T_{v-1}| = O(1) \; .$$

by hypotheses and Lemmas 1 and 2.

Let

$$f(\alpha, t) = \sum_{v=1}^{\infty} ( T_v - T_{v-1}) \sum_{n=v}^{\infty} c_{n-v} K_v(t)$$

$$= \left( \sum_{v=1}^{n} + \sum_{v=n+1}^{\infty} \right) ( T_v - T_{v-1}) Z_v(t)$$

$$= \chi_1 + \chi_2 \, , \text{ say.} \tag{4.5.1}$$

Now, as in the proof of (3.5.2) we have

$$|\chi_2| = O(1) \frac{P_\mu}{\mu} \, . \tag{4.5.2}$$

by hypotheses and Lemmas 2 and 12.

Next, we have

$$\chi_1 = -\sum_{v=1}^{n} w_v\, v\, \Delta z_{v-1}(t) + \sum_{v=1}^{n} w_v\, z_{v-1}(t) - n\, w_{n+1}\, z_n(t)$$

$$= L_1 + L_2 - n\, w_{n+1}\, z_n(t)$$

where, by Lemmas 3(ii) and 12 , we find that

$$L_1 = o(1) ;$$

and by applying Abel's transformation twice and by an appeal to Lemmas 3(ii) and 12 , we observe, in the same manner, that

$$L_2 = o(1) ;$$

and by Lemmas 1, 3(i) and 12 , $n\, w_n\, z_n(t) = o(1)$.

Hence, as in the proof of (3.5.3) , we get

(4.5.3) $$\chi_1 = L_1 + L_2 - n\, w_{n+1}\, z_n(t) = o(1).$$

Therefore, from (4.5.1) , (4.5.2) and (4.5.3) , we obtain

$$f(a, t) = o(1) + O(1)\, \frac{P_\mu}{\mu} , \text{ as } t \to 0.$$

Thus

$$\lim_{t\to o} \sup |f(\alpha, t)| \leq O(1) \frac{P_\mu}{\mu} .$$

$\mu$ being arbitrary large and $O(1)$ independent of $\mu$, we have

$$\lim_{t\to o} f(\alpha, t) = 0.$$

This completes the proof of Theorem 1.

## Chapter V

## ON THE TOTAL REGULARITY OF RIEMANN SUMMABILITY

5.1 <u>Definitions and Notations</u>. Let $\Sigma a_n$ be a given infinite series, with the sequence of its partial sums $\{s_n\}$, ($s_n = a_0 + a_1 + \dots + a_n$, $a_0 = 0$). Let $S_n^\alpha$ denote the n-th Cesàro-sum of order $\alpha$ ( $\alpha > -1$) of the sequence $\{s_n\}$, or the series $\Sigma a_n$, defined by :

$$(5.1.1) \qquad S_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha} a_\nu ,$$

where $A_n^\alpha$ is given by the identity :

$$(5.1.2) \qquad \sum_{n=0}^{\infty} A_n^\alpha x^n = (1-x)^{-1-\alpha} , \qquad ( |x| < 1 ).$$

Then the Cesaro transform of order $\alpha$ ( $\alpha > -1$ ) of the sequence $\{s_n\}$, or the series $\Sigma a_n$, is defined by :[1]

$$(5.1.3) \qquad s_n^\alpha = S_n^\alpha / A_n^\alpha .$$

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1) Hardy (1).

For $p > 0$, let us write

$$(5.1.4) \qquad f_p(x) = \left(\frac{\sin x}{x}\right)^p \quad (x \neq 0), \quad f_p(0) = 1.$$

The series $\Sigma a_n$ is said to be summable by Riemann-Cesaro method of order $p$ ( $p=1,2,3, \ldots$ ) and index $\alpha$ $(-1 \leq \alpha < p-1)$, or briefly summable $(R, p, \alpha)$, to sum $s$, if the series in

$$(5.1.5) \qquad F_p(\alpha,t) = (C_{p,\alpha})^{-1} t^{\alpha+1} \sum_{n=1}^{\infty} S_n^{\alpha} f_p(nt) ,$$

where

$$(5.1.6) \quad C_{p,\alpha} = \begin{cases} \dfrac{1}{\Gamma(\alpha+1)} \displaystyle\int_0^{\infty} u^{\alpha-p}(\sin u)^p du, & -1<\alpha<p-1 ; \\ \pi/2 & , \alpha=0, \ p=1 ; \\ 1 & , \alpha = -1 , \end{cases}$$

converges in some interval $0 < t < t_0$, and

$$\lim_{t \to 0} F_p(\alpha,t) = s .$$[1]

Under this definition the $(R,p,-1)$-transform, $F_p(-1,t)$, is identical with the $(R,p)$-transform, and the $(R,p,0)$-transform, $F_p(0,t)$ is identical with $(R_p)$-transform. Therefore, the summability methods $(R,p,-1)$ and $(R,p,0)$ are the same as summability methods $(R,p)$ and $(R_p)$ respectively.

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1) Hirokawa (1).

It is known that the method $(R, p, \alpha)$, $-1 \leq \alpha < p-1$, is regular for $p \geq 2$. [1]

Let $p > 0$, $0 \leq \lambda_0 < \lambda_1 < \dots < \lambda_n \to \infty$. If the series

$$R_\lambda^p(t) = \sum_{n=0}^{\infty} a_n f_p(\lambda_n t) \tag{5.1.7}$$

converges in some interval $0 < t < t_0$, and if

$$\lim_{t \to 0} R_\lambda^p(t) = s ,$$

then we say that the series $\Sigma a_n$ is summable $(R, \lambda_n, p)$ to sum s. [2] In the case when $\lambda_n = n$ and p is a positive integer, the summability $(R, \lambda_n, p)$ is the same as the summability $(R, p)$.

A method of summation or a transform associated with it, is said to be <u>regular</u> if it assigns to every convergent series

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1) Hirokawa (1).

2) This more general definition has been given by Burkill (1), for p = 1, 2, and by Burkill and Peterson(1) for p rational with odd denominator (which ensures that $f_p(x)$ is real), see also Russell (1).

its actual value. If it furthermore assigns the value $+\infty$ to every series which diverges to $+\infty$, it is said to be totally regular.[1]

We also write

$$(5.1.8) \qquad \Delta_n = \lambda_{n+1} / (\lambda_{n+1} - \lambda_n).$$

5.2 Introduction. Regarding the total regularity of the summability (R, 2), Lee[2] proved the following theorems.

Theorem A. The method (R, 2) is not totally regular. More precisely, given a sequence $\{W_n\}$ tending to $\infty$ arbitrarily slowly, there exists a series $\Sigma\, a_n$ with $|a_n| \leq W_n / n$, such that $\sum_{m=1}^{n} a_m \to \infty$, but

$$\liminf_{t \to 0} \sum_{n=1}^{\infty} a_n f_2 (n,t) = -\infty$$

Theorem B. If $a_n \to 0$, $a_n > -1/n$, then the method (R, 2) is totally regular.

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1) Hardy (1), §1.4 , page 10.

2) Lee (1)

From Theorem A, it follows that, in Theorem B, the condition on $a_n$ cannot be weakened, even in a two sided form.

Later on Hirokawa[1)] extended the above-mentioned theorems to the summability methods (R, p) by establishing the following results.

Theorem C. The method (R, p) is not totally regular when $p = 2, 3, \ldots$. More precisely, given a monotone increasing sequence $\{W_n\}$ tending to $+\infty$ such that

$W_n n^{-p} \to 0$, as $n \to \infty$, there exists a series $\sum_{n=1}^{\infty} a_n$

with $|a_n| \leq 2 W_n/n$ for all n, such that $\sum_{n=1}^{\infty} a_n = +\infty$

and

$$\liminf_{t \to 0+} \sum_{n=1}^{\infty} a_n f_p(nt) = -\infty.$$

Theorem D. Let $p = 1, 2, \ldots$. Suppose that

$a_n \geq -K/n$ ($n = 1, 2, 3, \ldots$; K ; a positive constant),

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1) Hirokawa (4)

$\sum_{n=1}^{\infty} a_n = +\infty$ and $\sum_{n=1}^{\infty} a_n f_p(nt)$ converges in $0 < t < t_o$.

Then

$$\lim_{t\to o} \sum_{n=1}^{\infty} a_n f_p(nt) = +\infty.$$

Concerning the total regularity of the methods $(R_p)$ and $(R, p, \alpha)$, Hirokawa[1] also proved the following interesting results.

Theorem E. The method $(R_{2p})$, $p=1, 2, \ldots$, is totally regular.

Theorem F. The method $(R_{2p+1})$, $p=1, 2, \ldots$, is not totally regular.

Theorem G. The method $(R, 2p, \alpha)$, $0 \leq \alpha < 2p$, $p=1, 2, \ldots$, is totally regular.

In the present chapter we establish two theorem in this direction. Theorem 1 concerns with the non-total regularity

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1) Hirokawa (4)

of the method $(R, 2p+1, \alpha)$ for certain values of p, while Theorem 2 deals with the total regularity of the method $(R, \lambda_n, p)$, for $p=1,2,\ldots$ , and with certain conditions on the sequence, $\lambda_n$ . These theorems generalize the corresponding results of Theorems F and D respectively .

5.3 We establish the following.

Theorem 1. *The method* $(R, 2p+1, \alpha)$, $0 \leq \alpha < 2p$, $p = 1,2,\ldots$ , *is not totally regular.*

Theorem 2. *Let* $p=1,2,\ldots$ , *and let* $0 \leq \lambda_0 < \lambda_1 < \ldots < \lambda_n \to \infty$, *such that*

$$\sum_{n=\nu}^{\infty} \frac{1}{\Lambda_n \lambda_n^p} = O\left(\frac{1}{\lambda_\nu^p}\right), \tag{5.3.1}$$

*and for every sequence* $\{t_\nu\}$ *of positive numbers tending to zero, and integers* $N_\nu$, $\lim_{\nu\to\infty} \lambda_{N_\nu} t_\nu = \pi$ . *If* $a_n \geq -K/\Lambda_n$ $(n = 1,2, \ldots$ ; $K$ ; *is a positive constant)* , *and the series*

$$\sum_{n=1}^{\infty} a_n f_p(\lambda_n t)$$

*converges, then* $\sum_{n=1}^{\infty} a_n = +\infty$ *implies*

$$\lim_{t\to 0} \sum_{n=1}^{\infty} a_n f_p(\lambda_n t) = +\infty.$$

5.4 We need the following lemmas for proving our theorems.

Lemma 1.[1] In order that a real sequence-to-sequence transformation :

$$(5.4.1) \qquad t_n = \sum_{m=0}^{n} c_{n,m} s_m$$

should be totally regular, it is necessary and sufficient that it should be regular and positive.

Lemma 2.[2] It is necessary, for a real regular transformation

$$(5.4.2) \qquad y(t) = \sum_{k=1}^{\infty} a_k(t)\, s_k ,$$

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1) Hurwitz (1); see also Hardy (1), Theorem 10, page 53. The Transform(5.4.1) is said to be positive if $c_{m,n} \geq 0$ for all m,n or at any rate for $n \geq n_0$ .

2) Hurwitz (1), Theorem 6.

*to be totally regular, that*

$$(5.4.3)\qquad \lim_{D(t)\to 0} \sum_{k=1}^{\infty} \Big[ |a_k(t)| - a_k(t) \Big] = 0.$$

*Remark.* In (5.4.2) and (5.4.3), $t$ is a variable ranging over some point set, $D(t)$ is a positive real function ; and the functions $a_k(t)$ are real, but not necessarily continuous.

*Lemma 3.*[1] *If* $p$ *is a positive integer and* $p > \alpha+1 > 0$, *then*

$$(5.4.4)\quad \lim_{t\to 0} t^{\alpha+1} \sum_{n=1}^{\infty} A_n^{\alpha} \left( \frac{\sin nt}{nt} \right)^p$$

$$= \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} u^{\alpha-p} (\sin u)^p \, du$$

$$= C_{p,\alpha} .$$

*5.5 Proof of Theorem 1 .* Since it is known that, for $\alpha \geq 0$, the Cesàro transform $s_n^{\alpha}$ is positive and regular , by Lemma 1, it is also totally regular, that is,

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1) Rangachari (1), Lemma 2; see also Hirokawa (1).

$$s_n \to s \quad \text{implies} \quad s_n^{\alpha} \to s\ ,$$

for s finite or infinite.

Now, by definition $(R,\ 2p+1,\ \alpha)$-transform is given by

$$F_{2p+1}(\alpha, t) = (C_{2p+1,\alpha})^{-1}\ t^{\alpha+1} \sum_{n=1}^{\infty} S_n^{\alpha}\ f_{2p+1}(nt)$$

$$= (C_{2p+1,\alpha})^{-1}\ t^{\alpha+1} \sum_{n=1}^{\infty} s_n^{\alpha}\ A_n^{\alpha}\ f_{2p+1}(nt).$$

This transforms a sequence $s_n^{\alpha}$ into a function $F_{2p+1}(\alpha, t)$ and is also real and regular.

In order to prove the theorem let us suppose, on the contrary, that the transform $F_{2p+1}(\alpha, t)$ is totally regular. Then by Lemma 2 it is necessary that,

$$\lim_{t\to 0} \sum_{n=1}^{\infty} A_n^{\alpha} \left[ |f_{2p+1}(nt)| - f_{2p+1}(nt) \right] = 0.$$

But, we observe that, by Lemma 3, for $0 \leq \alpha < 2p$,

$$\lim_{t\to 0} \sum_{n=1}^{\infty} A_n^{\alpha} \left[ \left| \frac{\sin nt}{nt} \right|^{2p+1} - \left( \frac{\sin nt}{nt} \right)^{2p+1} \right]$$

$$= \frac{1}{\Gamma(\alpha+1)} \left[ \int_0^{\infty} u^{\alpha-2p-1} \left| \sin u \right|^{2p+1} du - \int_0^{\infty} u^{\alpha-2p-1} (\sin u)^{2p+1} du \right] > 0$$

which leads to a contradiction. Hence the $(R, 2p+1, \alpha)$-transform is not totally regular.

This completes the proof of Theorem 1.

5.6 Proof of Theorem 2. To prove the theorem it is sufficient, under the hypothesis, to show that

$$\sum_{n=1}^{\infty} a_n \left( \frac{\sin \lambda_n t_v}{\lambda_n t_v} \right)^p \to \infty ,$$

for every sequence $\{t_v\}$ of positive numbers tending to zero.

Since, $\Sigma a_n = +\infty$, there exists an integer $N_1$ such that $a_n > 0$ when $n > N_1$ , and an integer $N_2 = N_2(G) > N_1$, corresponding to any large number $G$, such that $a_n > 0$ .

Now, by Abel's partial summation,

$$(5.6.1) \qquad \Sigma_v = \sum_{n=1}^{\infty} a_n f_p(\lambda_n t_v)$$

$$= \sum_{1}^{N_1} s_n \left[ f_p(\lambda_n t_v) - f_p(\lambda_{n+1} t_v) \right]$$

$$+ \sum_{n=N_1+1}^{N_2} s_n \left[ f_p(\lambda_n t_v) - f_p(\lambda_{n+1} t_v) \right]$$

$$+ \sum_{n=N_2+1}^{N_v-1} s_n \left[ f_p(\lambda_n t_v) - f_p(\lambda_{n+1} t_v) \right]$$

$$+ s_{N_v} f_p(\lambda_{N_v} t_v) + \sum_{n=N_v}^{\infty} a_n f_p(\lambda_n t_v)$$

$$= \Sigma_{v,1} + \Sigma_{v,2} + \Sigma_{v,3} + \Sigma_{v,4} + \Sigma_{v,5}, \text{ say.}$$

Now, since $f_p(x)$ is monotonic in $(0,\pi)$, we have

$$\Sigma_{v,3} = \sum_{n=N_2+1}^{N_v-1} s_n \left[ f_p(\lambda_n t_v) - f_p(\lambda_{n+1} t_v) \right]$$

$$\geq G \sum_{n=N_2+1}^{N_v-1} \left[ f_p(\lambda_n t_v) - f_p(\lambda_{n+1} t_v) \right]$$

$$= G \left[ f_p(\lambda_{N_2+1} t_v) - f_p(\lambda_{N_v} t_v) \right],$$

and hence

$$(5.6.2) \qquad \liminf_{v\to\infty} \Sigma_{v,3} \geq G\left[ f_p(o) - f_p(\pi) \right] = G.$$

and

$$\Sigma_{v,5} = \sum_{n=N_v}^{\infty} a_n f_p(\lambda_n t_v)$$

$$\geq - K \sum_{n=N_v}^{\infty} \frac{1}{\Lambda_n \lambda_n^p t_v^p}$$

$$= - \frac{K}{t_v^p} \sum_{n=N_v}^{\infty} \frac{1}{\Lambda_n \lambda_n^p}$$

$$\geq - K_2 \frac{1}{\lambda_{N_v}^p t_v^p},$$

by hypothesis, where $K_2$ is a positive constant, and so

(5.6.3) $$\liminf_{\nu \to \infty} \Sigma_{\nu,5} \geq - \frac{K_\rho}{n^p} .$$

we have further

(5.6.4) $$\Sigma_{\nu,2} \geq 0 , \quad \Sigma_{\nu,4} \geq 0$$

and

(5.6.5) $$\lim_{\nu \to \infty} \Sigma_{\nu,1} = 0 .$$

Hence combining (5.6.1) - (5.6.5), we obtain

$$\liminf_{\nu \to \infty} \Sigma_\nu \geq 0 ,$$

and consequently

$$\lim_{\nu \to \infty} \Sigma_\nu = + \infty .$$

This terminates the proof of Theorem 2.

## Chapter VI

## ON ABSOLUTE ABEL AND ABSOLUTE RIEMANN SUMMABILITY

6.1 <u>Definitions and Notations</u> : Let $\sum a_n$ be a given infinite series with the sequence of its partial sums $\{s_n\}$, where $s_n = a_0 + a_1 + \dots + a_n$, and let us write

$$(6.1.1) \qquad f(x) = \sum_{n=0}^{\infty} a_n x^n = (1-x) \sum_{n=0}^{\infty} s_n x^n .$$

If the series on the right of (6.1.1) converges for $0 \le x < 1$, and if $f(x) \in BV\,[0, 1)$, that is,

$$\int_0^1 |f'(x)|\, dx < \infty ,$$

then the series $\sum a_n$ is said to be absolutely summable by Abel's method, or simply summable $|A|$.[1]

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1) Whittaker (1), Prasad (1).

The series $\Sigma a_n$ is said to be absolutely summable (A) with index k $(k \geq 1)$, or simply summable $|A|_k$, if the series on the right of (6.1.1) converges for $0 \leq x < 1$, and

$$\int_0^1 (1-x)^{k-1} \, |f'(x)|^k \, dx < \infty .$$ [1]

Let us write

$$f_p(x) = \left(\frac{\sin x}{x}\right)^p , \; x \neq 0, \quad f_p(0) = 1.$$

The series $\Sigma a_n$ is said to be summable $|R, p|$, where p is a positive integer, if the series

$$(6.1.2) \qquad F_p(x) = \sum_{k=1}^{\infty} a_k \; f_p(kx)$$

is convergent for $x \in (0, \delta)$, $\delta > 0$ and $F_p(x) \in BV[0, \delta)$, that is,

$$\int_0^{\delta} |d( F_p(x) )| < \infty .$$

or,

$$\int_0^{\delta} \left|\frac{d}{dx}( F_p(x) )\right| dx < \infty .$$ [2]

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1) Flett (1), [illegible] (1).

2) Gesberg (1).

We say that the series $\Sigma a_n$ is summable $|R, p|_k$, where p is a positive integer, and $k \geq 1$, if the series (6.1.2) is convergent for $x \in [0, \delta)$, $\delta > 0$, and

$$\int_0^{\delta} x^{k-1} \left| \frac{d}{dx} ( F_p(x) ) \right|^k dx < \infty .$$

The method $|R, p|_1$ is the same as the method $|R, p|$. For $k > 1$ the methods $|R, p|$ and $|R, p|_k$ are independent.

Throughout this chapter, we use the following notations:

For $0 \leq r < 1$,

$$C(r, \theta) = 1/2 + \sum_{n=1}^{\infty} r^n \cos n\theta ;$$

$$S(r, \theta) = \sum_{n=1}^{\infty} r^n \sin n\theta .$$

6.2 <u>Introduction</u>. Recently, in a different context, Gesberg[1] proved the following theorem giving a relation between absolute Abel and absolute Riemann summability methods.

Theorem A. If a series $\Sigma a_n$ is summable $|R, p|$ and, if $a_n = O(1)$, then it is also summable $|A|$.

Since, for $k > 1$, the methods $|A|$ and $|R, p|$ are independent

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1) Gesberg (1) . -

of the methods $|A|_k$ and $|R, p|_k$ respectively, our object in this chapter is to extend Theorem A for the methods $|A|_k$ and $|R, p|_k$.

6.3 We establish the following theorem.

<u>Theorem.</u> <u>Let</u> $k \geq 1$. <u>If the series</u> $\Sigma a_n$ <u>is summable</u> $|R, p|_k$ <u>and, if</u> $s_n = O(1)$, <u>then it is also summable</u> $|A|_k$.

6.4 The following lemmas are needed for the proof of our theorem.

<u>Lemma</u> 1.[1] <u>Let</u> $v \geq 1$ <u>be an integer, then for</u> $0 < \theta \leq \pi$,

$$\text{(i)} \quad \frac{\partial^v}{\partial\theta^v}\left\{N(r,\theta)\right\} = O\left(r \left|(1-r)^2 + \theta^2\right|^{-(v+1)/2}\right);$$

and

$$\text{(ii)} \quad N(r,\theta) = \frac{r \sin\theta}{1-2r\cos\theta + r^2} = O\left[\frac{r\,\theta}{(1-r)^2 + \theta^2}\right].$$

<u>Lemma</u> 2. <u>Let</u> $v \geq 1$ <u>be an integer. Then, for</u> $0 \leq r < 1$, $k \geq 1$, <u>and</u> $0 < \theta \leq \pi$,

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1) cf. Gesberg(1), proof of Lemma 2.

(a) $$\int_0^1 (1-r)^{k-1} \left| \frac{\theta^v}{r} \frac{\partial^v}{\partial\theta^v} N(r,\theta) \right|^k dr = O(\theta^{-k}),$$

and

(b) $$\int_0^1 (1-r)^{k-1} \left| \frac{N(r,\theta)}{r} \right|^k dr = O(\theta^{-k}).$$

Proof (b) By Lemma 1(ii), we have

$$\int_0^1 (1-r)^{k-1} \frac{N(r,\theta)}{r}^k dr = O(1) \int_0^1 (1-r)^{k-1} \left| \frac{r\theta}{r[(1-r)^2+\theta^2]} \right|^k dr$$

$$= O(1)\theta^{-k} \int_0^1 (1-r)^{k-1} dr$$

$$= O(\theta^{-k}).$$

(a) Similarly, by Lemma 1(i), we have

$$\int_0^1 (1.r)^{k-1} \left| \frac{\theta^v}{r} . \frac{\partial^v}{\partial\theta^v} N(r,\theta) \right|^k dr$$

$$= O(1) \int_0^1 (1-r)^{k-1} \left\{ \frac{\theta^v}{[(1-r)^2 + \theta^2]^{(v+1)/2}} \right\}^k dr$$

$$= O(1) \left(\frac{\theta^v}{\theta^{v+1}}\right)^k \int_0^1 (1-r)^{k-1} dr$$

$$= O(\theta^{-k}).$$

Lemma 3. For $i > 1$ and $k > 1$,

$$(a)\quad \int_0^1 (1-r)^{k-1} \left| \sum_{j=0}^{\infty} r^{2(j+1)} - i\, r^{2i(j+1)} \right|^k dr = O(1),$$

and

$$(b)\quad \int_0^1 (1-r)^{k-1} \left| (1-r^2) \sum_{j=0}^{\infty} r^{2(j+1)} - i\, r^{2i(j+1)} \right|^k dr = O(1).$$

Proof. For $r < 1$, we have

$$(6.4.1)\quad \left| r^{2(k+1)} - i\, r^{2i(k+1)} \right| = \begin{cases} r^{2(k+1)} - i\, r^{2i(k+1)}, & \text{for } k \geq k_0, \\ i\, r^{2i(k+1)} - r^{2(k+1)}, & \text{for } k \leq k_0 \end{cases}$$

where

$$k_0 = \left[ - \frac{\log i}{2(i-1)\log r} \right] - 1.$$

Hence

$$L(r) = \sum_{k=0}^{\infty} \left| r^{2(k+1)} - i\, r^{2i(k+1)} \right|$$

$$= -\frac{1-r^{\rho(k_o+1)}}{1-r^{\rho}} + i\,\frac{1-r^{\rho i(k_o+1)}}{1-r^{\rho i}} + \frac{r^{\rho(k_o+1)}}{1-r^{\rho}} - i\,\frac{r^{\rho i(k_o+1)}}{1-r^{\rho}} .$$

It can be shown that

$$r^{\rho i(k_o+1)} = r^{\rho(k_o+1)} + O(1-r) ,$$

and thus

$$L(r) = (1-r^{\rho(k_o+1)})\left(-\frac{1}{1-r^{\rho}} + \frac{1}{1-r^{\rho i}}\right) + O(1),$$

Hence, by Minskowski's inequality, for $k > 1$, and any $t$ ( $0 \le t < 1$) ,

$$\int_0^t (1-r)^{k-1} \left|L(r)\right|^k dr$$

$$= O(1) \left[1+\left(\int_0^t (1-r)^{k-1} \left|\left\{1-r^{\rho(k_o+1)}\right\} \left(-\frac{1}{1-r^{\rho}}+\frac{1}{1-r^{\rho i}}\right)\right|^k dr\right)^{1/k}\right]$$

$$= O(1),$$

which is equivalent to proving (a). The evaluation of (b) can be done analogously with the help of equation which arises from (6.4.1),

$$\sum_{k=1}^{\infty} k \left| r^{2(k+1)} - i r^{2i(k+1)} \right|$$

$$= \sum_{k=1}^{k_0} k \left\{ i r^{2i(k+1)} - r^{2(k+1)} \right\} + \sum_{k=k_0+1}^{\infty} k \left\{ r^{2(k+1)} - i r^{2i(k+1)} \right\}$$

6.5 <u>Proof of the theorem</u>. Since, for $k=1$, the theorem is known, we proceed to prove the theorem for $k > 1$.

Let us consider two cases separately, namely, (I) when $p$ is an even integer and (II) when $p$ is an odd integer.

Case I : <u>Let</u> $p$ <u>be an even integer, that is</u>, $p = 2m$, for $m = 1, 2, \ldots$ . Since,

$$\sin^{2m} jx = 2^{-2m} \sum_{i=0}^{m} (-1)^{i} \binom{2m}{i} \cos 2(m-i)\, jx \, , \tag{6.5.1}$$

we have

$$F^{*}(x) = \frac{1}{2} \sum_{j=1}^{\infty} \frac{a_j}{j^{2m}} \sin^{2m} jx \tag{6.5.2}$$

$$= 2^{-2m} \sum_{j=1}^{\infty} \frac{a_j}{j^{2m}} \sum_{i=0}^{m} (-1)^{i} \binom{2m}{i} \cos 2(m-i) jx.$$

Due to the hypothesis; $a_n = O(1)$, the double series converges absolutely for $x \leq \pi$ . For,

$$(6.5.3) \qquad F(x) = 2^{-2m} \sum_{\ell=0}^{\infty} \left( \Sigma' \frac{a_j}{j^{2m}} \binom{2m}{i} \right) \cos 2\ell x$$

where $\Sigma'$ denotes the summation for each $\ell$, such value of $j$ and $i \leq m$ for which $j(m-i) = \ell$; then the series on the right hand side of (6.5.3) is absolutely convergent. Hence, the series on the right of (6.5.3) may be used as the Fourier series of the function $F^*(x)$. By virtue of this fact, for $0 \leq r < 1$, we have

$$\phi(x,r) = 2^{-2m} \sum_{j=1}^{\infty} \frac{a_j}{j^{2m}} \sum_{i=0}^{m} (-1)^i \binom{2m}{i} r^{2(m-i)j} \cos 2(m-i)\, jx$$

$$= 2^{-2m} \sum_{\ell=0}^{\infty} \Sigma' \frac{a_j}{j^{2m}} (-1)^i \binom{2m}{i} r^{2\ell} \cos 2\ell x$$

$$= \frac{1}{\pi} \int_0^{\pi} K(r, \theta - x) \; F^*(\theta)\, d\theta.$$

It is evident that, for $0 \leq r < 1$, the equation

$$(6.5.4) \qquad \phi(x,r) = \frac{1}{\pi} \int_{-\pi}^{\pi} K(r, \theta - x)\, F(\theta)\, d\theta$$

can be differentiated with respect to $x$ '$2m$' times. Therefore, after differentiating $m$-times and putting $x=0$, we get

$$(6.5.5) \qquad h(r) = \left[\frac{d^m}{dx^m}\,\phi(x,r)\right]_{x=0}$$

$$= \sum_{j=1}^{\infty} a_j \sum_{i=0}^{m} (-1)^i \binom{m}{i} r^{2(m-i)j}\,(m-i)^{2m}$$

$$\cdot \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\frac{\partial^{2m}}{\partial\theta^{2m}} K(r,\theta)\right) F^*(\theta)\,d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi} \theta^{2m}\,\frac{\partial^{2m}}{\partial\theta^{2m}}\left\{K(r,\theta)\right\}\frac{F^*(\theta)}{\theta^{2m}}\,d\theta$$

$$= -\frac{2}{\pi}\left[\left(\int_0^{\theta} t^{2m}\left(\frac{\partial^{2m}}{\partial t^{2m}} K(r,t)\right)dt\right) F_{2m}(\theta)\right]_0^{\pi}$$

$$+ \frac{2}{\pi}\int_0^{\pi}\left[\int_0^{\theta} t^{2m}\left(\frac{\partial^{2m}}{\partial t^{2m}} K(r,t)\,dt\right)\right] d\,F_{2m}(\theta)$$

$$= \frac{2}{\pi}\int_0^{\pi}\left[\int_0^{\theta} t^{2m}\left(\frac{\partial^{2m}}{\partial t^{2m}} K(r,t)\right)dt\right] d\,F_{2m}(\theta)\ ,$$

since, for $\theta = \pi$, $F_p(\theta) = 0$, and for $\theta = 0$, the integral

$$\int_0^{\theta} t^{2m}\left(\frac{\partial^{2m}}{\partial t^{2m}} K(r,t)\right) dt = 0.$$

Now, integrating by parts ,

$$\int_0^\theta t^{2m} \left\{ \frac{\partial^{2m}}{\partial t^{2m}} M\ (r,t) \right\} dt$$

$$= \sum_{i=1}^{2m} (-1)^{i+1}\ \theta^{2m-i+1}\ \frac{\partial^{2m-i}}{\partial \theta^{2m-i}} M\ (r,\theta) + \int_0^\theta M(r,t)\,dt,$$

Substituting this expression in (6.5.5), we get

$$(6.5.6)\quad H(r) = \frac{2}{\pi} \sum_{i=1}^{2m} \int_0^\pi \theta^{2m-i+1} \left(\frac{\partial^{2m-i}}{\partial \theta^{2m-i}} M(r,\theta)(-1)^i\, dF_{2m}(\theta)\right.$$

$$+ \frac{2}{\pi} \int_0^\pi \left( \int_0^\theta M(r,t)\ dt\right)\ d\ F_{2m}(\theta)$$

As the integral on the right of (6.5.6) and their partial derivatives are continuous for $0 \leq r < 1$, $0 \leq \theta < \pi$, differentiating (6.5.6) with respect to r (under the integral sign), we get

$$(6.5.7)\quad H'(r) = \frac{2}{\pi} \sum_{i=1}^{2m} \int_0^\pi (-1)^i\ \theta^{2m-i+1}\ \frac{\partial^{2m-i+1}}{\partial r \partial \theta^{2m-i}} M(r,\theta)$$

$$+ \frac{2}{\pi} \int_0^\pi \left( \frac{d}{dr} \int_0^\theta M\ (r,t)\ dt \right)\ d\ F_{2m}(\theta).$$

Now, since, for $1 \leq i \leq 2m$,

$$
(6.5.8)\quad \begin{cases} \dfrac{\partial^{2m-i+1}}{\partial r\,\partial\theta^{2m-i}} M(r,\theta) = \dfrac{1}{r}\dfrac{\partial^{2m-i+1}}{\partial\theta^{2m-i+1}} \left\{ N(r,\theta) \right\} ; \\[2ex] \dfrac{d}{dr} \displaystyle\int_0^{\pi} M\,(r,t)\,dt = \dfrac{1}{\pi} N(r,0)\ , \end{cases}
$$

(6.5.7) becomes

$$
H'(r) = \frac{2}{\pi} \sum_{i=1}^{2m} \int_0^{\pi} (-1)^i \frac{\partial^{2m-i+1}}{r} \frac{\partial^{2m-i+1}}{\partial\theta^{2m-i+1}} \left\{ N(r,\theta) \right\} d\,F_{2m}(\theta)
$$

$$
+ \frac{2}{\pi} \int_0^{\pi} \frac{1}{r}\ N(r,\theta)\ d\,F_{2m}(\theta).
$$

Now, by Minkowski's inequality,

$$
I = \int_0^1 (1-r)^{k-1} \left| H'(r) \right|^k dr
$$

$$
\leq \left\{ \int_0^1 (1-r)^{k-1} \frac{2^k}{\pi^k} \left| \sum_{i=1}^{2m} \int_0^{\pi} (-1)^i \theta^{2m-i+1} \frac{1}{r} \frac{\partial^{2m-i+1}}{\partial\theta^{2m-i+1}} N(r,\theta)\, dF_{2m}(\theta) \right|^k dr \right\}^{1/k}
$$

$$
+ \left\{ \int_0^1 (1-r)^{k-1} \left| \frac{2}{\pi} \int_0^{\pi} \frac{N(r,\theta)}{r}\ d\,F_{2m}(\theta) \right|^k dr \right\}^{1/k}
$$

$$
= \frac{2}{\pi} \left( I_1 \right)^{1/k} + \frac{2}{\pi} \left( I_2 \right)^{1/k} ,
$$

say, where

$$I_1 = \int_0^1 (1-r)^{k-1} \left| \sum_{i=1}^{2m} \int_0^\pi (-1)^i \frac{\partial^{2m-i+1}}{r} \frac{\partial^{2m-i+1}}{\partial\theta^{2m-i+1}} H(r,\theta)\, dF_{2m}(\theta) \right|^k dr$$

and

$$I_2 = \int_0^1 (1-r)^{k-1} \left| \int_0^\pi \frac{H(r,\theta)}{r}\, d\, F_{2m}(\theta) \right|^k dr.$$

Therefore, in order to show that, under the hypothesis of the theorem,

$$(6.5.9) \qquad \int_0^1 (1-r)^{k-1} \left| H'(r) \right|^k dr = O(1),$$

it is sufficient to show that,

$$(6.5.10) \quad I_{1,i} = \int_0^1 (1-r)^{k-1} \left| \int_0^\pi (-1)^i \frac{\partial^{2m-i+1}}{r} \frac{\partial^{2m-i+1}}{\partial\theta^{2m-i+1}} H(r,\theta)\, dF_{2m}(\theta) \right|^k dr = O(1)$$

for $1 \leq i \leq 2m$;

and

$$(6.5.11) \qquad I_2 = O(1).$$

<u>Proof of</u> (6.5.10). By Hölder's inequality and Lemma 2(a), for $1 \leq i \leq 2m$, we have

$$I_{1,i} = \int_0^1 (1-r)^{k-1} \left| \int_0^\pi (-1)^i \theta^{2m-i+1} \frac{1}{r} \frac{\partial^{2m-i+1}}{\partial \theta^{2m-i+1}} N(r,\theta) \frac{dF_{2m}(\theta)}{d\theta} d\theta \right|^k dr$$

$$= O(1) \int_0^1 (1-r)^{k-1} dr \left\{ \int_0^\pi \left| \frac{\theta^{2m-i+1}}{r} \frac{\partial^{2m-i+1}}{\partial \theta^{2m-i+1}} N(r,\theta) \right| \left| \frac{dF_{2m}(\theta)}{d\theta} \right|^k d\theta \right\}^{1/k}$$

$$= O(1) \int_0^1 (1-r)^{k-1} dr \int_0^\pi \theta^{-1} \left| \frac{\theta^{2m-i+1}}{r} \frac{\partial^{2m-i+1}}{\partial \theta^{2m-i+1}} N(r,\theta) \right|^k$$

$$\left| \frac{d\, F_{2m}(\theta)}{d\theta} \right|^k d\theta \; \left( \int_0^\pi \theta^{1/k-1} \right)^{k-1}$$

$$= O(1) \int_0^\pi \theta^{-1} \left| \frac{d}{d\theta} \{ F_{2m}(\theta) \} \right|^k d\theta \int_0^1 (1-r)^{k-1} \left| \frac{\theta^{2m-i+1}}{r} \right.$$

$$\left. \frac{\partial^{2m-i+1}}{\partial \theta^{2m-i+1}} N(r,\theta) \right|^k dr$$

$$= O(1) \int_0^\pi \theta^{-1} \left| \frac{d}{d\theta} \{ F_{2m}(\theta) \} \right|^k d\theta \, . \, \theta^{-k}$$

$$= O(1) \int_0^\pi \theta^{-k-1} \left| \frac{d}{d\theta} \{ F_{2m}(\theta) \} \right|^k d\theta$$

$$= O(1) \int_0^\pi \theta^{k-1} \left| \frac{d}{d\theta} \{ F_{gm}(\theta) \} \right|^k d\theta$$

$$= O(1),$$

by hypothesis.

<u>Proof of</u> (6.5.11). By Hölder's inequality and Lemma 2(b), we have

$$I_2 = \int_0^1 (1-r)^{k-1} \left| \int_0^\pi \frac{1}{r} N(r,\theta) \frac{d}{d\theta} \{ F_{gm}(\theta) \} d\theta \right|^k dr$$

$$= O(1) \int_0^1 (1-r)^{k-1} \left( \int_0^\pi \left| \frac{1}{r} N(r,\theta) \right| \left| \frac{d}{d\theta} \{ F_{gm}(\theta) \} \right| d\theta \right)^k dr$$

$$= O(1) \int_0^1 (1-r)^{k-1} dr \int_0^\pi \theta^{-1} \left| \frac{1}{r} N(r,\theta) \right|^k \left| \frac{d}{d\theta} \{ F_{gm}(\theta) \} \right|^k d\theta \left( \int_0^\pi \theta^{1/k-1} \right)^{k-1}.$$

$$= O(1) \int_0^\pi \theta^{-1} \left| \frac{d}{d\theta} \{ F_{gm}(\theta) \} \right|^k d\theta \int_0^1 (1-r)^{k-1} \left| \frac{1}{r} N(r,\theta) \right|^k dr$$

$$= O(1) \int_0^\pi \theta^{-1} \left| \frac{d}{d\theta} \{ F_{gm}(\theta) \} \right|^k d\theta . \ \theta^{-k}$$

$$= O(1) \int_0^\pi \theta^{-k-1} \left| \frac{d}{d\theta} \{ F_{gm}(\theta) \} \right|^k d\theta$$

$$= O(1) \int_0^{\pi} \theta^{k-1} \left| \frac{d}{d\theta} \{ F_{2m}(\theta) \} \right|^k d\theta,$$

$$= O(1),$$

by hypothesis.

Now, for $0 \leq r < 1$, let us write

$$G(r) = \sum_{j=0}^{\infty} j\, a_j\, r^{2j} .$$

Thus, in order to prove the theorem, under the hypothesis, it is enough to show that

$$(6.5.12) \qquad \int_0^1 (1-r)^{k-1} \left| \frac{G(r)}{r} \right|^k dr = O(1),$$

since this is equivalent to $|A|_k$ - summability of the series $\Sigma a_n$ .

<u>Proof of</u> (6.5.12). We see that, by (6.5.5), we have

$$H'(r) = \frac{2}{r} \sum_{i=0}^{m} (-1)^i \binom{2m}{i} (m-i)^{2m+1} G(r^{(m-i)}).$$

Hence,

$$\int_0^1 (1-r)^{k-1} \left| \frac{G(r)}{r} \right|^k dr$$

$$= \left(\frac{2^m}{\lfloor 2m}\right)^k \int_0^1 (1-r)^{k-1} \left| \frac{G(r)}{r} \sum_{i=0}^{m} (-1)^i \binom{2m}{i} (m-i)^{2m} \right|^k dr$$

$$\left(\text{since } \sum_{i=0}^{m} (-1)^i \binom{2m}{i} (m-i) = \frac{\lfloor 2m}{2^m}\right)$$

$$= O(1) \int_0^1 (1-r)^{k-1} \left| \left[ H'(r) + \sum_{i=0}^{m} (-1)^i \binom{2m}{i} \left\{ (m-i) G(r^{(m-i)}) - G(r) \right\} \right] \right|^k dr$$

$$= O(1) \left( \int_0^1 (1-r)^{k-1} \left| H'(r) \right|^k dr \right)^{1/k}$$

$$+ \left( \int_0^1 (1-r)^{k-1} \left| \sum_{i=0}^{m} (-1)^i \binom{2m}{i} \left\{ (m-i) G(r^{(m-i)}) - G(r) \right\} \right|^k dr \right)^{1/k}$$

Thus, by virtue of (6.5.9), in order to prove (6.5.12), it is enough to show that, for $0 \leq i \leq m$,

$$(6.5.13) \qquad \int_0^1 (1-r)^{k-1} \left| i(G(r^i) - G(r) \right|^k dr = O(1).$$

<u>Proof of</u> (6.5.13). We see that, for $0 \leq i \leq m$, applying Abel's transformation,

$$\left| i\, G(r^{i}) - G(r) \right| = \left| \sum_{j=0}^{\infty} k\, a_k \,(i\, r^{2ij} - r^{2j}) \right|$$

$$= \left| \sum_{j=0}^{\infty} \Big( \sum_{=1}^{j} u_\ell \Big)\, i \left\{ j\, r^{2ij} - (j+1)\, r^{2i(j+1)} \right\} \right.$$

$$\left. - \left\{ j\, r^{2j} - (j+1)\, r^{2(j+1)} \right\} \right|$$

$$\leq \left| \sum_{j=0}^{\infty} a_j \left\{ r^{2i(j+1)} - i\, r^{2i(j+1)} \right\} \right| + (1-r^{2}) \left| \sum_{j=0}^{\infty} a_j\, j\, (r^{2j} - i\, r^{2ij}) \right|$$

$$= O(1) \left[ \sum_{j=0}^{\infty} \left| r^{2(j+1)} - i\, r^{2i(j+1)} \right| + (1-r^{2}) \sum_{j=0}^{\infty} j \left| r^{2j} - i\, r^{2ij} \right| \right],$$

And, by applying Lemma 3, we obtain

$$\int_0^1 (1-r)^{k-1} \left| i\, G(r^{i}) - G(r) \right|^{k} dr = O(1).$$

This terminates the proof of the theorem in the case when p is an even integer.

Case II : Let p be an odd integer, that is, $p = 2m+1$, for $m = 0, 1, 2, \ldots$ . Then

$$\sin^{2m+1} x = 2^{-2m} \sum_{i=0}^{m} (-1)^{i} \binom{2m+1}{i} \sin\left[2(m-i)+1\right] x$$

and considering as in Case I, it is easy to get :

$$\Phi(x,r) = 2^{-2m} \sum_{j=1}^{\infty} a_j \, j^{-(2m+1)} \sum_{i=0}^{m} (-1)^{i} \binom{2m+1}{i} \sin\left[2(m-i)+1\right] x$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} N(r,\theta-x) \; F_1^{*}(\theta)\,d\theta,$$

where

$$F_1^{*}(\theta) = \sum_{k=1}^{\infty} a_k \, k^{-(2m+1)} \sin^{2m+1} k\theta .$$

Further proof will be the same as in the case I, if we replace '2m' by '2m+1' .

This terminates the proof of the theorem.

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# A P P E N D I X

INSTITUTO MATEMATICO
'GUIDO CASTELNUOVO'
Universita Di Roma

Roma, October 16, 1972

Prof. V.K.Parasher,
Opp. City Post Office,
ALIGARH, India

Dear Prof. Parasher,

Your work entitled 'On $(N, p_n)$ and $(K, 1, \alpha)$ summability methods' has been duly received by 'Redazione dei Rendiconti di Matematica', and we are glad to communicate you that it has been accepted for publication in our Journal.

Presumably your article will appear in the second or in the third issue of 1973.

Please, excuse us very much for our letter of October 3, 1972 and for the trouble caused to you.

Yours sincerely,

sd/-xxxxxxxxxx

Rendiconti Di Matematica

Attested
Z. U. Ahmad
(Supervisor)

INDIAN NATIONAL SCIENCE ACADEMY
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Ref.No.EP/270JM/1509 Dated 3/8/1973

Paper entitled On a Relation between ($N$, $p_n$) and ($R$, 1, $\alpha$)-summability (with Dr. Z.U.Ahmad)

Dear Shri Parashar,

I am glad to inform you that the above paper which was communicated by Prof. R.S.Mishra, FNA has been accepted for publication.

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